

A Joint Normal-Binary (Probit) Model for High-Dimensional Longitudinal Data

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Supplementary Materials

A Proofs of the conditional distribution of a subvector of the continuous response

Consider the most general case, where we have the conditional distribution of a subvector of the continuous response given a subvector of the binary responses and a subvector of the continuous responses. In Delporte et al. (2022), the expected value is equal to

$$\begin{aligned}
 E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{Y}}_{ci}^b = \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = \mathbf{1}] &= \frac{e^{-0.5G_i}}{(2\pi)^{\frac{n_b}{2}} f(\tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = \mathbf{1})} \frac{\sqrt{|\mathbf{E}_i| |\mathbf{T}_i|}}{\sqrt{|\mathbf{V}_i| |\mathbf{B}_i| |\mathbf{E}_i^{bb}|}} \\
 &\quad \Phi(\tilde{\mathbf{X}}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}; \mathbf{F}_i; \mathbf{T}_i) \\
 &\quad \left\{ \left((\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}_1)^a + \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}_1)^b) \right) \right. \\
 &\quad + \left((\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \right) \\
 &\quad \times \left. \left(\mathbf{T}_i \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix} + \mathbf{F}_i \right) \right\},
 \end{aligned} \tag{A.1}$$

This expression can be simplified by proving that

$$\frac{e^{-0.5G_i}}{(2\pi)^{\frac{n_b}{2}} f(\tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = 1)} \frac{\sqrt{|\mathbf{E}_i| |\mathbf{T}_i|}}{\sqrt{|\mathbf{V}_i| |\mathbf{B}_i| |\mathbf{E}_i^{bb}|}} \Phi(\tilde{\mathbf{X}}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}; \mathbf{F}_i; \mathbf{T}_i) = 1 \quad (\text{A.2})$$

First consider,

$$\begin{aligned} \mathbf{T}_i^{-1} &= (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})' (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b + \mathbf{B}_i^{-1} - (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i (\mathbf{H}_i' \mathbf{B}_i^{-1}) \\ \mathbf{B}_i^{-1} \mathbf{H}_i &= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \end{aligned}$$

Further, define

$$\begin{aligned} \mathbf{M}_a &= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{aa})^{-1} \\ \mathbf{M}_b &= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{bb})^{-1} \end{aligned}$$

Since $\boldsymbol{\Sigma}_i = \sigma_i^2 \mathbf{I}$ and the fact that $\tilde{\mathbf{Z}}_{ci}$ and $\tilde{\mathbf{Z}}_{bi}$ are design matrices

$$(\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} = \mathbf{M}_b \mathbf{E}^{bb} + \mathbf{M}_a \mathbf{E}^{ab}$$

As a result,

$$\begin{aligned} \mathbf{T}_i^{-1} &= \mathbf{M}_b \mathbf{E}^{bb} \mathbf{M}_b' + \mathbf{M}_b \mathbf{E}^{ab} \mathbf{M}_a' + \mathbf{M}_a \mathbf{E}^{ab} \mathbf{M}_b' + \mathbf{M}_a \mathbf{E}^{ab} (\mathbf{E}_i^{bb})^{-1} \mathbf{E}^{ba} \mathbf{M}_a' + \mathbf{B}_i^{-1} - \\ &\quad (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i (\mathbf{H}_i' \mathbf{B}_i^{-1}) \\ &= \mathbf{M}_b \mathbf{E}^{bb} \mathbf{M}_b' + \mathbf{M}_b \mathbf{E}^{ab} \mathbf{M}_a' + \mathbf{M}_a \mathbf{E}^{ab} \mathbf{M}_b' + \mathbf{M}_a \mathbf{E}_i^{ab} (\mathbf{E}_i^{-1})^{bb} \mathbf{E}_i^{ba} \mathbf{M}_a' - \\ &\quad \mathbf{M}_a \mathbf{E}_i^{ab} (\mathbf{E}_i^{-1})^{ba} ((\mathbf{E}_i^{-1})^{aa})^{-1} (\mathbf{E}_i^{-1})^{ab} \mathbf{E}_i^{ba} \mathbf{M}_a' + \mathbf{B}_i^{-1} - (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i (\mathbf{H}_i' \mathbf{B}_i^{-1}), \end{aligned}$$

where we used the inverse of partitioned matrices,

$$(\mathbf{E}_i^{bb})^{-1} = (\mathbf{E}_i^{-1})^{bb} - (\mathbf{E}_i^{-1})^{ba} ((\mathbf{E}_i^{-1})^{aa})^{-1} (\mathbf{E}_i^{-1})^{ab},$$

Further,

$$\begin{aligned} \mathbf{T}_i^{-1} &= \mathbf{I} - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}'_{bi} + \mathbf{M}_a \left(\mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} \mathbf{E}_i^{ba} - \mathbf{E}_i^{aa} \right) \mathbf{M}'_a \\ &= \mathbf{I} - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}'_{bi} - \mathbf{M}_a \left((\mathbf{E}_i^{-1})^{aa} \right)^{-1} \mathbf{M}'_a, \end{aligned}$$

where we used $\mathbf{B}_i^{-1} = \mathbf{I} - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}'_{bi}$ and $(\mathbf{H}'_i \mathbf{B}_i^{-1}) = [\mathbf{M}_b \ \mathbf{M}_a]$ and the inverse of a partitioned matrix.

If we re-substitute $\mathbf{M}_a = -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}'^{a'}_{ci} (\Sigma_i^{aa})^{-1}$

$$\begin{aligned} \mathbf{T}_i^{-1} &= \mathbf{I} - \tilde{\mathbf{Z}}_{bi} \left[\mathbf{K}_i + \mathbf{K}_i \tilde{\mathbf{Z}}'^{a'}_{ci} (\Sigma_i^{aa})^{-1} \left((\mathbf{E}_i^{-1})^{aa} \right)^{-1} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}^a_{ci} \mathbf{K}_i \right] \tilde{\mathbf{Z}}'_{bi} \\ &= \mathbf{I} - \tilde{\mathbf{Z}}_{bi} \left[\mathbf{W}_i \right] \tilde{\mathbf{Z}}'_{bi}. \end{aligned}$$

Next,

$$\mathbf{W}_i^{-1} = \mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}'^{a'}_{ci} (\Sigma_i^{aa})^{-1} \left[(\mathbf{E}_i^{-1})^{aa} + (\mathbf{K}_i \tilde{\mathbf{Z}}'^{a'}_{ci} (\Sigma_i^{aa})^{-1})' \tilde{\mathbf{Z}}'^{a'}_{ci} (\Sigma_i^{aa})^{-1} \right]^{-1} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}^a_{ci}$$

and

$$\begin{aligned} \mathbf{E}_i^{-1} &= \Sigma_i^{-1} + \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} \left(\mathbf{K}_i \tilde{\mathbf{Z}}'_{bi} \mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i - (\mathbf{D}^{-1} + \tilde{\mathbf{Z}}'_{ci} \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci})^{-1} \right) \tilde{\mathbf{Z}}'_{ci} \Sigma_i^{-1} \\ &= \Sigma_i^{-1} + \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} \left(\mathbf{K}_i \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i + \right. \\ &\quad \left. \mathbf{K}_i \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} (\mathbf{D}_i^{-1} + \tilde{\mathbf{Z}}'_{ci} \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci})^{-1} \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i - (\mathbf{D}^{-1} + \tilde{\mathbf{Z}}'_{ci} \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci})^{-1} \right) \tilde{\mathbf{Z}}'_{ci} \Sigma_i^{-1} \\ &= \Sigma_i^{-1} + \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} \left(-\mathbf{K}_i \right) \tilde{\mathbf{Z}}'_{ci} \Sigma_i^{-1}, \end{aligned}$$

where we used $\mathbf{B}_i = \mathbf{I} + \tilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi})^{-1} \tilde{\mathbf{Z}}'_{bi}$ and $\tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} = \mathbf{K}_i^{-1} - \mathbf{D}^{-1} -$

$\tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci}$. As a result,

$$\begin{aligned}
 \mathbf{W}_i^{-1} &= \mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \left[(\Sigma_i^{aa})^{-1} - (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} + \right. \\
 &\quad \left. (\mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1})' \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \right]^{-1} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a \\
 &= \mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \left[(\Sigma_i^{aa})^{-1} \right]^{-1} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a \\
 &= \mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a \\
 &= \mathbf{D}^{-1} + \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} + \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} - \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a \\
 &= \mathbf{D}^{-1} + \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} + \tilde{\mathbf{Z}}_{ci}^b (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b
 \end{aligned}$$

As a consequence,

$$\mathbf{T}_i^{-1} = \mathbf{I} - \tilde{\mathbf{Z}}_{bi} \left[\mathbf{D}^{-1} + \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} + \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \right]^{-1} \tilde{\mathbf{Z}}_{bi}', \quad (\text{A.3})$$

which equals $(\mathbf{B}_i^*)^{-1}$, the inverse of the $\mathbf{B}_i$ matrix of the joint density $f(\tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi})$.

Next, consider

$$\begin{aligned}
 \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i &= -\mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{T}_i \left((\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \right. \\
 &\quad \left. (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} (\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b) \right)
 \end{aligned} \quad (\text{A.4})$$

Next,

$$\begin{aligned}
(\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} &= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} \\
&= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \\
&\quad \left\{ -\tilde{\mathbf{Z}}_{ci}^a [-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci}]^{-1} \tilde{\mathbf{Z}}_{ci}^{b'} \right. \\
&\quad \left. [\Sigma_i^{bb} - \tilde{\mathbf{Z}}_{ci}^b (-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci})^{-1} \tilde{\mathbf{Z}}_{ci}^{b'}]^{-1} \right\} \\
&= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} + \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a [-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci}]^{-1} \tilde{\mathbf{Z}}_{ci}^{b'} \\
&\quad [(\Sigma_i^{bb})^{-1} - (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1}] \\
&= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} + \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1},
\end{aligned}$$

where $\mathbf{K}_i^* = (\mathbf{D}^{-1} + \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} + \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b)^{-1}$ and we made the following substitutions

$$\begin{aligned}
\tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b &= (\mathbf{K}_i^*)^{-1} + \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1}, \\
(\mathbf{K}_i)^{-1} + \tilde{\mathbf{Z}}_{ci}' (\Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci}) &= -\tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1}.
\end{aligned}$$

Next, consider

$$\mathbf{Z}_{ci}^{a'} (\Sigma_i^{aa})^{-1} \tilde{\mathbf{Z}}_{ci}^a = \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} - \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b$$

As a consequence,

$$\begin{aligned}
(\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} &= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i (\tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} - \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b) \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \\
&= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} - \\
&\quad \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i (\mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1} - (\mathbf{K}_i^*)^{-1} + \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} + \mathbf{D}^{-1}) \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \\
&= -\tilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^*) \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1}
\end{aligned}$$

Hence,

$$-\mathbf{T}_i (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} \tilde{\mathbf{y}}_{ci}^b = -\mathbf{H}_i^* \tilde{\mathbf{y}}_{ci}^b, \quad (\text{A.5})$$

where equals $\mathbf{H}_i^*$ equals the $\mathbf{H}_i$ matrix of the joint density $f(\tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi})$.

Next, we will rewrite the second part of (A.4)

$$\begin{aligned}
\mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i + \mathbf{H}_i^* \widetilde{\mathbf{y}}_{ci}^b &= -\mathbf{B}_i \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{T}_i \left((\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \right. \\
&\quad \left. (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} (-(\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b) \right) \\
&= -\mathbf{B}_i \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{B}_i^* (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&= -\mathbf{B}_i \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&= -\mathbf{B}_i \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \left(\boldsymbol{\Sigma}_i - \widetilde{\mathbf{Z}}_{ci} [-\mathbf{K}_i^{-1} + \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci}]^{-1} \widetilde{\mathbf{Z}}_{ci}' \right) \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&= -\mathbf{B}_i \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \left(\widetilde{\mathbf{Z}}_{ci} [-\mathbf{K}_i^{-1} + \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci}]^{-1} \widetilde{\mathbf{Z}}_{ci}' \right) \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&= -\mathbf{B}_i \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} [-\widetilde{\mathbf{Z}}_{bi}' \widetilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1}]^{-1} \widetilde{\mathbf{Z}}_{ci}' \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b,
\end{aligned}$$

where we rewrote $\widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci} = \mathbf{K}_i^{-1} - \widetilde{\mathbf{Z}}_{bi}' \widetilde{\mathbf{Z}}_{bi}$

Next,

$$\begin{aligned}
(\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b &= \mathbf{E}_i^{bb} (\mathbf{V}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b + \mathbf{E}_i^{ba} (\mathbf{V}_i^{-1})^{ab} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&\quad + \mathbf{E}_i^{bb} (\mathbf{V}_i^{-1})^{ba} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a + \mathbf{E}_i^{ba} (\mathbf{V}_i^{-1})^{aa} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a \\
&= \left(\Sigma_i^{bb} - \widetilde{\mathbf{Z}}_{ci}^b [-\mathbf{K}_i^{-1} + \widetilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \widetilde{\mathbf{Z}}_{ci}]^{-1} \widetilde{\mathbf{Z}}_{ci}^{b'} \right) (\mathbf{V}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b + \\
&\quad \left(\Sigma_i^{ba} - \widetilde{\mathbf{Z}}_{ci}^b [-\mathbf{K}_i^{-1} + \widetilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \widetilde{\mathbf{Z}}_{ci}]^{-1} \widetilde{\mathbf{Z}}_{ci}^{a'} \right) (\mathbf{V}_i^{-1})^{ab} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b + \\
&\quad \left(\Sigma_i^{bb} - \widetilde{\mathbf{Z}}_{ci}^b [-\mathbf{K}_i^{-1} + \widetilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \widetilde{\mathbf{Z}}_{ci}]^{-1} \widetilde{\mathbf{Z}}_{ci}^{b'} \right) (\mathbf{V}_i^{-1})^{ba} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a + \\
&\quad \left(\Sigma_i^{ba} - \widetilde{\mathbf{Z}}_{ci}^b [-\mathbf{K}_i^{-1} + \widetilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \widetilde{\mathbf{Z}}_{ci}]^{-1} \widetilde{\mathbf{Z}}_{ci}^{a'} \right) (\mathbf{V}_i^{-1})^{aa} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a
\end{aligned}$$

As a consequence,

$$\begin{aligned}
H_i \widetilde{X}_{ci} \beta - F_i + H_i^* \widetilde{y}_{ci}^b &= -B_i \widetilde{Z}_{bi} K_i \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{X}_{ci} \beta - \\
& B_i^* \widetilde{Z}_{bi} [-\widetilde{Z}_{bi}' \widetilde{Z}_{bi} - D^{-1}]^{-1} \widetilde{Z}_{ci}' V_i^{-1} \widetilde{X}_{ci} \beta - \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{b'} (V_i^{-1})^{bb} (\widetilde{X}_{ci} \beta)^b + \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \widetilde{Z}_{ci}^b \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{b'} (V_i^{-1})^{bb} (\widetilde{X}_{ci} \beta)^b + \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \widetilde{Z}_{ci}^b \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{a'} (V_i^{-1})^{ab} (\widetilde{X}_{ci} \beta)^b - \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{b'} (V_i^{-1})^{ba} (\widetilde{X}_{ci} \beta)^a + \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \widetilde{Z}_{ci}^b \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{b'} (V_i^{-1})^{ba} (\widetilde{X}_{ci} \beta)^a + \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \widetilde{Z}_{ci}^b \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{a'} (V_i^{-1})^{aa} (\widetilde{X}_{ci} \beta)^a \\
&= -B_i \widetilde{Z}_{bi} K_i \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{X}_{ci} \beta - \\
& B_i^* \widetilde{Z}_{bi} \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}' V_i^{-1} \widetilde{X}_{ci} \beta + \\
& B_i^* \widetilde{Z}_{bi} \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{b'} (V_i^{-1})^{bb} (\widetilde{X}_{ci} \beta)^b + \\
& B_i^* \widetilde{Z}_{bi} \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{a'} (V_i^{-1})^{ab} (\widetilde{X}_{ci} \beta)^b + \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{a'} (V_i^{-1})^{ab} (\widetilde{X}_{ci} \beta)^b - \\
& B_i^* \widetilde{Z}_{bi} \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{b'} (V_i^{-1})^{ba} (\widetilde{X}_{ci} \beta)^a + \\
& B_i^* \widetilde{Z}_{bi} \left[-K_i^{-1} + \widetilde{Z}_{ci}' \Sigma_i^{-1} \widetilde{Z}_{ci} \right]^{-1} \widetilde{Z}_{ci}^{a'} (V_i^{-1})^{aa} (\widetilde{X}_{ci} \beta)^a + \\
& B_i^* \widetilde{Z}_{bi} K_i^* \widetilde{Z}_{ci}^{a'} (V_i^{-1})^{aa} (\widetilde{X}_{ci} \beta)^a
\end{aligned}$$

where we substituted $\widetilde{Z}_{ci}^{b'} \Sigma_i^{-1} \widetilde{Z}_{ci}^b = (K_i^*)^{-1} - \widetilde{Z}_{ci}^{b'} \widetilde{Z}_{ci}^b - D^{-1}$.

Next, consider

$$\widetilde{Z}_{ci}^{a'} (V_i^{-1})^{ab} (\widetilde{X}_{ci} \beta)^b + \widetilde{Z}_{ci}^{a'} (V_i^{-1})^{aa} (\widetilde{X}_{ci} \beta)^a = \widetilde{Z}_{ci}^{a'} (V_i^{-1})^a (\widetilde{X}_{ci} \beta)$$

and

$$\tilde{\mathbf{Z}}_{ci}^{b'}(\mathbf{V}_i^{-1})^{ba}(\tilde{\mathbf{X}}_{ci}\boldsymbol{\beta})^a + \tilde{\mathbf{Z}}_{ci}^{b'}(\mathbf{V}_i^{-1})^{bb}(\tilde{\mathbf{X}}_{ci}\boldsymbol{\beta})^b = \tilde{\mathbf{Z}}_{ci}^{b'}(\mathbf{V}_i^{-1})^b(\tilde{\mathbf{X}}_{ci}\boldsymbol{\beta})$$

As a result,

$$\begin{aligned} \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i + \mathbf{H}_i^* \tilde{\mathbf{y}}_{ci}^b &= -\mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\ &\quad \mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi} \left[-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} \right]^{-1} \tilde{\mathbf{Z}}_{ci}' \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\ &\quad \mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi} \left[-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} \right]^{-1} \tilde{\mathbf{Z}}_{ci}^{b'} (\mathbf{V}_i^{-1})^b \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\ &\quad \mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi} \left[-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} \right]^{-1} \tilde{\mathbf{Z}}_{ci}^{a'} (\mathbf{V}_i^{-1})^a \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\ &\quad \mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{a'} (\mathbf{V}_i^{-1})^a \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\ &= -\mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{a'} (\mathbf{V}_i^{-1})^a \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\ &= -\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \tilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi})^{-1} \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\ &\quad \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{a'} (\mathbf{V}_i^{-1})^a \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\ &\quad \tilde{\mathbf{Z}}_{bi} ((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi})^{-1} \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{a'} (\mathbf{V}_i^{-1})^a \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \end{aligned}$$

where we have rewritten $\mathbf{B}_i = \mathbf{I} + \tilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi})^{-1} \tilde{\mathbf{Z}}_{bi}'$ and $\mathbf{B}_i^* = \mathbf{I} + \tilde{\mathbf{Z}}_{bi} ((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi})^{-1} \tilde{\mathbf{Z}}_{bi}'$.

Next, the substitution of

$$\begin{aligned} \mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} &= \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} + \mathbf{D}^{-1}, \\ (\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} &= \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1}, \\ \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} &= \mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} - \mathbf{D}^{-1}, \\ \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} &= (\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b - \mathbf{D}^{-1}, \end{aligned}$$

results in

$$\begin{aligned}
\mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i + \mathbf{H}_i^* \widetilde{\mathbf{y}}_{ci} &= -\widetilde{\mathbf{Z}}_{bi} (\widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci} + \mathbf{D}^{-1})^{-1} \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \widetilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^* + \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{bi}' \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^*) \widetilde{\mathbf{Z}}_{ci}^{a'} \left((\boldsymbol{\Sigma}_i + \widetilde{\mathbf{Z}}_{ci} \mathbf{D} \widetilde{\mathbf{Z}}_{ci}')^{-1} \right)^a \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\
&= -\widetilde{\mathbf{Z}}_{bi} (\widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci} + \mathbf{D}^{-1})^{-1} \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \widetilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^* + \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{bi}' \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^*) \widetilde{\mathbf{Z}}_{ci}^{a'} \left((\boldsymbol{\Sigma}_i + \widetilde{\mathbf{Z}}_{ci} \mathbf{D} \widetilde{\mathbf{Z}}_{ci}')^{-1} \right)^{ab} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \widetilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^* + \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{bi}' \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^*) \widetilde{\mathbf{Z}}_{ci}^{a'} \left((\boldsymbol{\Sigma}_i + \widetilde{\mathbf{Z}}_{ci} \mathbf{D} \widetilde{\mathbf{Z}}_{ci}')^{-1} \right)^{aa} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}.
\end{aligned}$$

Next, we substitute

$$\begin{aligned}
\left((\boldsymbol{\Sigma}_i + \widetilde{\mathbf{Z}}_{ci} \mathbf{D} \widetilde{\mathbf{Z}}_{ci}')^{-1} \right)^{ab} &= -(\boldsymbol{\Sigma}_i^{-1})^{aa} \widetilde{\mathbf{Z}}_{ci}^a (\mathbf{D}^{-1} + \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci}) \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb}, \\
\left((\boldsymbol{\Sigma}_i + \widetilde{\mathbf{Z}}_{ci} \mathbf{D} \widetilde{\mathbf{Z}}_{ci}')^{-1} \right)^{aa} &= (\boldsymbol{\Sigma}_i^{-1})^{aa} - (\boldsymbol{\Sigma}_i^{-1})^{aa} \widetilde{\mathbf{Z}}_{ci}^a (\mathbf{D}^{-1} + \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci}) \widetilde{\mathbf{Z}}_{ci}^{a'} (\boldsymbol{\Sigma}_i^{-1})^{aa}, \\
\mathbf{K}_i^* + \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{bi}' \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* &= (\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1})^{-1}, \\
\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b &= \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1} - \mathbf{D}^{-1}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i + \mathbf{H}_i^* \widetilde{\mathbf{y}}_{ci} &= -\widetilde{\mathbf{Z}}_{bi} (\widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci} + \mathbf{D}^{-1})^{-1} \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\
&\quad \widetilde{\mathbf{Z}}_{bi} (\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1})^{-1} \widetilde{\mathbf{Z}}_{ci}^{a'} (\boldsymbol{\Sigma}_i^{-1})^{aa} \widetilde{\mathbf{Z}}_{ci}^a (\widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci} + \mathbf{D}^{-1})^{-1} \\
&\quad \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b + \\
&\quad \widetilde{\mathbf{Z}}_{bi} (\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1})^{-1} \widetilde{\mathbf{Z}}_{ci}^{a'} (\boldsymbol{\Sigma}_i^{-1})^{aa} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a - \\
&\quad \widetilde{\mathbf{Z}}_{bi} (\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1})^{-1} \widetilde{\mathbf{Z}}_{ci}^{a'} (\boldsymbol{\Sigma}_i^{-1})^{aa} \widetilde{\mathbf{Z}}_{ci}^{a'} (\widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{Z}}_{ci} + \mathbf{D}^{-1})^{-1} \\
&\quad \widetilde{\mathbf{Z}}_{ci}^{a'} (\boldsymbol{\Sigma}_i^{-1})^{aa} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a \\
&= -\widetilde{\mathbf{Z}}_{bi} (\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1})^{-1} \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b,
\end{aligned}$$

where we rewrote $\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b + \widetilde{\mathbf{Z}}_{ci}^{a'} (\boldsymbol{\Sigma}_i^{-1})^{aa} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a = \widetilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}$.

Further,

$$\begin{aligned}
\mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i + \mathbf{H}_i^* \widetilde{\mathbf{y}}_{ci} &= - \left(\widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* - \widetilde{\mathbf{Z}}_{bi} ((\mathbf{K}_i^*)^{-1})^{-1} + \right. \\
&\quad \left. \widetilde{\mathbf{Z}}_{bi} [(\mathbf{K}_i^*)^{-1} - \widetilde{\mathbf{Z}}_{bi}' \widetilde{\mathbf{Z}}_{bi}]^{-1} \right) \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&= - \left(\left[\mathbf{I} + \widetilde{\mathbf{Z}}_{bi} [(\mathbf{K}_i^*)^{-1} - \widetilde{\mathbf{Z}}_{bi}' \widetilde{\mathbf{Z}}_{bi}]^{-1} \widetilde{\mathbf{Z}}_{bi}' \right] \widetilde{\mathbf{Z}}_{bi} \right. \\
&\quad \left. \left[\widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \widetilde{\mathbf{Z}}_{ci}^b + \mathbf{D}^{-1} + \widetilde{\mathbf{Z}}_{bi}' \widetilde{\mathbf{Z}}_{bi} \right]^{-1} \right) \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&= - \mathbf{B}_i^* \widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{bb})^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
&= \mathbf{H}_i^* (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b.
\end{aligned} \tag{A.6}$$

The combination of (A.3), (A.5) and (A.6) results in the following equality

$$\Phi((\mathbf{X}_{bi} \boldsymbol{\beta})^b - \mathbf{H}_i^* (\widetilde{\mathbf{y}}_{ci}^b - (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b); \mathbf{B}_i^*) = \Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}, \mathbf{F}_i, \mathbf{T}_i). \tag{A.7}$$

Next consider,

$$\begin{aligned}
G_i &= \left(\widetilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' (\mathbf{E}_i^{bb})^{-1} \left(\widetilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) - \mathbf{F}_i' \mathbf{T}_i^{-1} \mathbf{F}_i + \\
&\quad (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}) - (\mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{E}_i (\mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}) \\
&= - \left(\widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\widetilde{\mathbf{y}}_{ci}^b - (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b) - (\mathbf{B}_i^*)^{-1} \mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \right)' \mathbf{T}_i \\
&\quad \left(\widetilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \widetilde{\mathbf{Z}}_{ci}^{b'} (\widetilde{\mathbf{y}}_{ci}^b - (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b) - (\mathbf{B}_i^*)^{-1} \mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \right) \\
&\quad + \left(\widetilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' (\mathbf{E}_i^{bb})^{-1} \left(\widetilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) \\
&\quad + (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}) - (\mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{E}_i (\mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}).
\end{aligned}$$

Now consider the terms from the latter equation where $\tilde{\mathbf{y}}_{ci}^b$ occurs twice:

$$\begin{aligned}
& - \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right) + \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \tilde{\mathbf{y}}_{ci}^b \\
& = - \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right) + \\
& \quad \tilde{\mathbf{y}}_{ci}^{b'} \left((\mathbf{E}_i^{-1})^{bb} - (\mathbf{E}_i^{-1})^{ba} ((\mathbf{E}_i^{-1})^{aa})^{-1} (\mathbf{E}_i^{-1})^{ab} \right) \tilde{\mathbf{y}}_{ci}^b \\
& = \tilde{\mathbf{y}}_{ci}^{b'} \left\{ - (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi} \mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi}' \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} + (\boldsymbol{\Sigma}_i^{-1})^{bb} - (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} - \right. \\
& \quad (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} + \\
& \quad \left. (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \left(-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \right)^{-1} \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} \right\} \tilde{\mathbf{y}}_{ci}^b \\
& = \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b + \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \left\{ -\mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}' \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} ((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi})^{-1} \tilde{\mathbf{Z}}_{bi}' \right) \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* - \mathbf{K}_i - \right. \\
& \quad \left. \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \mathbf{K}_i + \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \left(-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \right)^{-1} \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \mathbf{K}_i \right\} \\
& \quad \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b,
\end{aligned}$$

where we first used the general inverse of block matrices

$$(\mathbf{E}_i^{bb})^{-1} = (\mathbf{E}_i^{-1})^{bb} - (\mathbf{E}_i^{-1})^{ba} ((\mathbf{E}_i^{-1})^{aa})^{-1} (\mathbf{E}_i^{-1})^{ab}$$

and next substituted $\mathbf{B}_i^* = \mathbf{I} + \tilde{\mathbf{Z}}_{bi} ((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi})^{-1} \tilde{\mathbf{Z}}_{bi}'$.

Further, we repeatedly use $\tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} = \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} - \mathbf{K}_i^{-1} + \mathbf{K}_i^{-1}$, which results in the

following

$$\begin{aligned}
& - \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right) + \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \tilde{\mathbf{y}}_{ci}^b \\
& = \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b + \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \left\{ \mathbf{K}_i^* - \left((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \right)^{-1} - \left(\mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{ci}^{a'} (\boldsymbol{\Sigma}_i^{-1})^{aa} \tilde{\mathbf{Z}}_{ci}^a \right)^{-1} \right\} \\
& \quad \tilde{\mathbf{Z}}_{ci}^b (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \\
& = \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b + \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \left\{ \mathbf{K}_i^* - \left((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \right)^{-1} - \mathbf{K}_i^* \right\} \tilde{\mathbf{Z}}_{ci}^b (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \\
& = \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b - \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \left(\mathbf{D}_i^{-1} + \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \right)^{-1} \tilde{\mathbf{Z}}_{ci}^b (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \\
& = \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{V}_i^*)^{-1} \tilde{\mathbf{y}}_{ci}^b,
\end{aligned}$$

where $(\mathbf{V}_i^*)^{-1}$ equals the inverse of the $\mathbf{V}_i$ matrix of the joint density $f(\tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi})$.

Next, consider the terms where $\tilde{\mathbf{y}}_{ci}^b$ occurs once, at the start of the term,

$$\begin{aligned}
& \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i (\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b + \\
& \quad \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i (\mathbf{B}_i^*)^{-1} \mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& = \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i (\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& \quad - \tilde{\mathbf{y}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}' \mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci} \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\
& \quad + \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \right)^{ba} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& \quad + \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \right)^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& = \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i (\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& \quad - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} \right)^b \mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \right)^b \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\
& = \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i (\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& \quad - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} \right)^b \mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \right)^b \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\
& = \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i (\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& \quad - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left((\mathbf{V}_i - \mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \mathbf{V}_i) \mathbf{H}_i' \mathbf{B}_i^{-1} \right)^b \mathbf{H}_i \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\
& \quad + \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \right)^b \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \\
& = \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{y}}_{ci}^b \right)' \mathbf{T}_i (\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b - \tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{E}_i^{bb})^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \\
& = -\tilde{\mathbf{y}}_{ci}^{b'} (\mathbf{V}_i^*)^{-1} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b,
\end{aligned}$$

where we have rewritten

$$\begin{aligned}
(\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b &= \left(\mathbf{I} - \mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \right)^{ba} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a + \\
&\quad \left(\mathbf{I} - \mathbf{V}_i \mathbf{H}_i' (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i')^{-1} \mathbf{H}_i \right)^{bb} (\widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b
\end{aligned}$$

where we implemented the results from the previous proofs.

Next consider,

$$\begin{aligned} \left(\mathbf{V}_i \mathbf{H}'_i (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}'_i)^{-1} \mathbf{H}_i \right)^b &= \tilde{\mathbf{Z}}_{ci}^b \mathbf{D} \tilde{\mathbf{Z}}'_{bi} (\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi})^{-1} \mathbf{H}_i \\ (\mathbf{E}_i^{bb})^{-1} &= (\boldsymbol{\Sigma}_i^{-1})^{bb} - (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \end{aligned}$$

As a consequence,

$$\begin{aligned} &\left(\mathbf{V}_i \mathbf{H}'_i (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}'_i)^{-1} \mathbf{H}_i \right)^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{V}_i \mathbf{H}'_i (\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}'_i)^{-1} \mathbf{H}_i \right)^b \\ &= \left(\tilde{\mathbf{Z}}_{ci}^b \mathbf{D} \tilde{\mathbf{Z}}'_{bi} (\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi})^{-1} \mathbf{H}_i \right)' \left((\boldsymbol{\Sigma}_i^{-1})^{bb} - (\boldsymbol{\Sigma}_i^{-1})^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\boldsymbol{\Sigma}_i^{-1})^{bb} \right) \\ &\quad \left(\tilde{\mathbf{Z}}_{ci}^b \mathbf{D} \tilde{\mathbf{Z}}'_{bi} (\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi})^{-1} \mathbf{H}_i \right) \\ &= \mathbf{H}'_i \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \right)^{-1} \tilde{\mathbf{Z}}_{bi} \mathbf{D} \left((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1} \right) \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \right)^{-1} \mathbf{H}_i \\ &\quad - \mathbf{H}'_i \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \right)^{-1} \tilde{\mathbf{Z}}_{bi} \mathbf{D} \left((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1} \right) \mathbf{K}_i^* \\ &\quad \left((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1} \right) \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \right)^{-1} \mathbf{H}_i \\ &= \mathbf{H}'_i \mathbf{H}_i - \mathbf{H}'_i \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \right)^{-1} \mathbf{H}_i - \mathbf{H}'_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}'_{bi} \mathbf{H}_i \\ &= \mathbf{H}'_i \mathbf{H}_i - \mathbf{H}'_i \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \right)^{-1} \mathbf{H}_i - \mathbf{H}'_i \left(\tilde{\mathbf{Z}}_{bi} (\mathbf{K}_i^*)^{-1} \tilde{\mathbf{Z}}'_{bi} + \mathbf{I} - \mathbf{I} \right) \mathbf{H}_i \\ &= - \mathbf{H}'_i \left(\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}'_{bi} \right)^{-1} \mathbf{H}_i + \mathbf{H}'_i (\mathbf{B}_i^*)^{-1} \mathbf{H}_i \\ &= - \mathbf{H}'_i \mathbf{H}_i + \mathbf{H}'_i \tilde{\mathbf{Z}}_{bi} \left(\mathbf{D}^{-1} + \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} \right)^{-1} \mathbf{H}_i + \mathbf{H}'_i (\mathbf{B}_i^*)^{-1} \mathbf{H}_i \\ &= \mathbf{H}'_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} + \mathbf{H}'_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}'_{bi} \mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} + \mathbf{H}'_i \tilde{\mathbf{Z}}_{bi} \left(\mathbf{D}^{-1} + \tilde{\mathbf{Z}}'_{bi} \tilde{\mathbf{Z}}_{bi} \right)^{-1} \tilde{\mathbf{Z}}'_{bi} \mathbf{H}_i + \\ &\quad \mathbf{H}'_i (\mathbf{B}_i^*)^{-1} \mathbf{H}_i, \end{aligned}$$

[illegible]

where we rewrote the following matrices

$$\begin{aligned} \mathbf{V}_i^{-1} &= \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} \mathbf{K}_i \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \\ &\quad - \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} \mathbf{K}_i \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \left(\mathbf{K}_i^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \right)^{-1} \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \boldsymbol{\Sigma}_i^{-1} \\ \mathbf{E}_i &= \boldsymbol{\Sigma}_i - \tilde{\mathbf{Z}}_{ci} \left(-\mathbf{K}_i^{-1} + \tilde{\mathbf{Z}}_{ci} \boldsymbol{\Sigma}_i^{-1} \tilde{\mathbf{Z}}_{ci} \right)^{-1} \tilde{\mathbf{Z}}_{ci}' \end{aligned}$$

As a result,

$$\begin{aligned} & - \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' \mathbf{T}_i (\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}') (\boldsymbol{\Sigma}_i^{-1})^{bb} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b - \\ & \quad \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' \mathbf{T}_i (\mathbf{B}_i^*)^{-1} \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \\ & \quad \left((\mathbf{B}_i^*)^{-1} \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \right)' \mathbf{T}_i \left(\tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' (\boldsymbol{\Sigma}_i^{-1})^{bb} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) - \\ & \quad \left((\mathbf{B}_i^*)^{-1} \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \right)' \mathbf{T}_i \left((\mathbf{B}_i^*)^{-1} \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \right) + (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i^{-1} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}) - \\ & \quad (\mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{E}_i (\mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}) + (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^{b'} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}) \\ & \quad = (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^{b'} (\mathbf{V}_i^*)^{-1} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \end{aligned}$$

Hence,

$$G_i = \left(\tilde{\mathbf{y}}_{ci}^{b'} - (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' (\mathbf{V}_i^*)^{-1} \left(\tilde{\mathbf{y}}_{ci}^{b'} - (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) \quad (\text{A.8})$$

Next, consider

$$\begin{aligned}
\frac{|\mathbf{E}_i||\mathbf{T}_i||\mathbf{V}_i^*|}{|\mathbf{V}_i||\mathbf{B}_i||\mathbf{E}_i^{bb}|} &= \frac{|(\mathbf{E}_i^{bb})^{-1}||\mathbf{V}_i^*|}{|\mathbf{V}_i||\mathbf{E}_i^{-1}||\mathbf{B}_i||\mathbf{T}_i^{-1}|} \\
&= \frac{|(\Sigma_i^{bb})^{-1} - (\Sigma_i^{bb})^{-1}\tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1}||\Sigma_i^{bb} + \tilde{\mathbf{Z}}_{ci}^b \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'}|}{|\mathbf{V}_i||\mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{H}_i + \mathbf{V}_i^{-1}||\mathbf{B}_i||\mathbf{T}_i^{-1}|} \\
&= \frac{|\mathbf{I} - (\Sigma_i^{bb})^{-1}\tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} + (\Sigma_i^{bb})^{-1}\tilde{\mathbf{Z}}_{ci}^b \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} - (\Sigma_i^{bb})^{-1}\tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1}\tilde{\mathbf{Z}}_{ci}^b \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'}|}{|\mathbf{V}_i \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{H}_i + \mathbf{I}_n||\mathbf{B}_i||\mathbf{T}_i^{-1}|} \\
&= |\mathbf{I} - (\Sigma_i^{bb})^{-1}\tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}^{b'} + \Sigma_i^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} + \Sigma_i^{bb} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \left((\mathbf{K}_i^*)^{-1} - \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} - \mathbf{D}^{-1} \right) \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'}| \\
&\quad \frac{1}{|\mathbf{I}_p + \mathbf{B}_i^{-1} \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i' ||\mathbf{B}_i||\mathbf{T}_i^{-1}|} \\
&= \frac{|\mathbf{I} + (\Sigma_i^{bb})^{-1}\tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'}|}{|\mathbf{B}_i + \mathbf{H}_i \mathbf{V}_i \mathbf{H}_i' ||\mathbf{T}_i^{-1}|} \\
&= \frac{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|}{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{bi}' ||\mathbf{T}_i^{-1}|} \\
&= \frac{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|}{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{bi}' ||\mathbf{I} - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|} \\
&= \frac{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|}{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{bi}' - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}' - \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{bi}' \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|} \\
&= \frac{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|}{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{bi}' (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|} \\
&= \frac{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{bi}' - \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}' - \tilde{\mathbf{Z}}_{bi} \mathbf{D} \left((\mathbf{K}_i^*)^{-1} - \mathbf{D}^{-1} - \tilde{\mathbf{Z}}_{ci}' (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \right) \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|}{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|} \\
&= \frac{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|}{|\mathbf{I} + \tilde{\mathbf{Z}}_{bi} \mathbf{D} \tilde{\mathbf{Z}}_{ci}^{b'} (\Sigma_i^{bb})^{-1} \tilde{\mathbf{Z}}_{ci}^b \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}'|}, \\
&= 1,
\end{aligned} \tag{A.9}$$

where the Sylvester identity $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$ is repeatedly used.

When we combine the results of (A.7), (A.8) and (A.9) in (A.1), the expected value

simplifies to

$$\begin{aligned}
E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{Y}}_{ci}^b = \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = 1] &= \left((\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}_1)^a \right. \\
&\quad \left. + \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}_1)^b) \right) \\
&\quad + \left((\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \right) \\
&\quad \times \left(\mathbf{T}_i \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix} + \mathbf{F}_i \right),
\end{aligned}$$

In addition, if we consider the special case of (A.1), where we only condition on the binary response, the expected value simplifies to

$$\begin{aligned}
E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{Y}}_{bi} = 1] &= \mathbf{E}_i (\mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{F}_i) \\
&\quad + \mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{T}_i \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix}.
\end{aligned}$$

B Prediction and confidence intervals for conditional expected values

2.1 Conditional distribution of the the continuous response given the binary responses

The prediction interval of the conditional expected value is composed of the second central moment and the standard errors of the transformed parameters. More specifically, the 95% prediction interval can be computed with the following general formula

$$\begin{aligned} & \left[E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1] - 1.96 \sqrt{E\left[(\tilde{\mathbf{Y}}_{ci} - E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1])^2\right] + \frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \text{Var}(\hat{\boldsymbol{\theta}}) \frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}}, \right. \\ & \left. E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1] + 1.96 \sqrt{E\left[(\tilde{\mathbf{Y}}_{ci} - E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1])^2\right] + \frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \text{Var}(\hat{\boldsymbol{\theta}}) \frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}} \right], \end{aligned}$$

where $\boldsymbol{\theta}$ denotes the parameter vector.

The second central moment is derived by Delporte et al. (2022) but can be simplified using (A.2). The expression is as follows

$$\begin{aligned} & E\left[(\tilde{\mathbf{Y}}_{ci} - E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1])(\tilde{\mathbf{Y}}_{ci} - E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1])'\right] \\ &= \mathbf{E}_i + \mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i^{-1} \mathbf{E}_i \\ & \quad + \mathbf{E}_i \mathbf{H}' \mathbf{B}_i^{-1} \left(\mathbf{N} + \mathbf{J} \mathbf{J}_i' \right) \mathbf{B}_i^{-1} \mathbf{H} \mathbf{E}_i + \\ & \quad \mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} \mathbf{J}' \mathbf{B}_i^{-1} \mathbf{H} \mathbf{E}_i + \mathbf{E}_i \mathbf{H}' \mathbf{B}_i^{-1} \mathbf{J} (\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i^{-1} \mathbf{E}_i \\ & \quad - E(\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1) E(\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = 1)', \end{aligned} \tag{B.1}$$

where $\mathbf{J}$ is the expected value of the truncated multivariate normal density, and $\mathbf{N}$ is the second central moment of the latter density. They are defined as follows:

$$\begin{aligned}
\mathbf{J} &= \mathbf{T}_i \left[-F_1(a_1) \quad -F_2(a_2) \quad \dots \quad -F_{\tilde{p}_i}(a_{\tilde{p}_i}) \right] + \mathbf{F}_i, \\
\mathbf{a} &= \widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} + \mathbf{H}\widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}, \\
\varphi(\mathbf{x}) &= \begin{cases} \frac{\phi(\mathbf{x}, \mathbf{F}_i, \mathbf{T}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} + \mathbf{H}\widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}, \mathbf{F}_i, \mathbf{T}_i)}, & \text{for } \mathbf{x} \leq \widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} + \mathbf{H}\widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}, \\ 0, & \text{otherwise.} \end{cases}, \\
F_i(x_i) &= \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_{i-1}} \int_{-\infty}^{a_{i+1}} \dots \int_{-\infty}^{a_{\tilde{p}_i}} \varphi(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{\tilde{p}_i}) dx_{\tilde{p}_i} \dots dx_{i+1} dx_{i-1} \dots dx_1, \\
N_{i,j} &= T_{i,j} + \sum_{k=1}^{\tilde{p}_i} T_{i,i,k} \frac{-T_{i,j,k} a_k F_k(a_k)}{T_{i,k,k}} + \sum_{k=1}^{\tilde{p}_i} T_{i,i,k} \sum_{q \neq k} \left(T_{i,j,q} - \frac{T_{i,k,q} T_{i,j,k}}{T_{i,k,k}} \right) \\
&\quad \cdot -F_{k,q}(a_k, a_q) - J_i J_k, \\
F_{k,q}(x, y) &= \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_{k-1}} \int_{-\infty}^{a_{k+1}} \dots \int_{-\infty}^{a_{q-1}} \int_{-\infty}^{a_{q+1}} \dots \int_{-\infty}^{a_{\tilde{n}_i}} \phi(x, y, \mathbf{x}_{-k,-q}) d\mathbf{x}_{-k,-q}.
\end{aligned}$$

The derivative of the expected value with respect to a coefficient β_{c2} of a predictor X_{c2} of the continuous response vector equals

$$\begin{aligned}
\frac{\partial E[\widetilde{\mathbf{Y}}_{ci} | \widetilde{\mathbf{y}}_{bi} = \mathbf{1}]}{\partial \beta_{c2}} &= \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}'_{c2i} + \mathbf{E}_i \mathbf{H}'_i \mathbf{B}_i^{-1} \mathbf{T}_i \mathbf{B}_i^{-1} \mathbf{H}_i \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}'_{bi} \\
&\quad + \mathbf{E}_i \mathbf{H}'_i \mathbf{B}_i^{-1} \frac{\boldsymbol{\nu} - \boldsymbol{\lambda} \mathbf{T}_i \left[-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p) \right]}{\Phi(\mathbf{o}, \mathbf{T}_i)}
\end{aligned} \tag{B.2}$$

with

$$\begin{aligned}
\boldsymbol{\lambda} &= \sum_{k=1}^{\tilde{p}_i} (\mathbf{H}_i \widetilde{\mathbf{X}}_{c2i} - \mathbf{T}_i \cdot \mathbf{B}_i^{-1} \mathbf{H}_i \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{c2i})_k \phi \left[(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} + \mathbf{H}_i \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta} - \mathbf{F}_i)_k, \mathbf{T}_{i,kk} \right] \\
&\quad \times \Phi \left[(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} + \mathbf{H}_i \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta} - \mathbf{F}_i)_{-k}; \mathbf{T}_{i,-k|k} \right], \\
\boldsymbol{\nu} &= \sum_{k=1}^{\tilde{p}_i} (\mathbf{H}_i \widetilde{\mathbf{X}}_{c2i} - \mathbf{T}_i \cdot \mathbf{B}_i^{-1} \mathbf{H}_i \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{c2i})_k g_k(o_k), \\
g_k(x_k) &= \int_{-\infty}^{o_1} \dots \int_{-\infty}^{o_{i-1}} \int_{-\infty}^{o_{i+1}} \dots \int_{-\infty}^{o_{\tilde{p}_i}} [x_1 \dots x_{k-1} o_k x_{k+1} \dots x_{\tilde{p}_i}]' \phi([x_1 \dots x_{k-1} o_k x_{k+1} \dots x_{\tilde{p}_i}]', \mathbf{T}_i) d\mathbf{x}_{-k}, \\
\mathbf{o} &= \widetilde{\mathbf{X}}_{2i}\boldsymbol{\beta} + \mathbf{H}_i \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta} - \mathbf{F}_i
\end{aligned}$$

Further, $\mathbf{T}_i$ is partitioned as

$$\mathbf{T}_i = \begin{bmatrix} \mathbf{T}_{11}^{(k)} & \mathbf{T}_{c2}^{(k)} \\ \mathbf{T}_{2c}^{(k)} & T_{kk} \end{bmatrix},$$

and $\mathbf{T}_{-k|k}$ is defined as

$$\mathbf{T}_{-k|k} = \mathbf{T}_{11}^{(k)} - \mathbf{T}_{c2}^{(k)} T_{kk}^{-1} \mathbf{T}_{2c}^{(k)}.$$

Next, for a coefficient β_{b2} of a predictor X_{b2} of the binary response vector the derivative is the following

$$\frac{E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = \mathbf{1}]}{\partial \beta_{b2}} = \mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} \frac{\xi - \Omega \cdot \mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]}{\Phi(\mathbf{o}, \mathbf{T}_i)},$$

where

$$\begin{aligned} \Omega &= \sum_{k=1}^{\tilde{p}} \tilde{\mathbf{X}}_{b2ik}' \phi[(\tilde{\mathbf{X}}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i)_k, \mathbf{T}_{i,kk}] \Phi[(\tilde{\mathbf{X}}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{F}_i)_{-k}; \mathbf{T}_{i,-k|k}], \\ \xi &= \sum_{k=1}^{\tilde{p}_i} \tilde{\mathbf{X}}_{b2ik}' g_k(o_k). \end{aligned}$$

The derivative of the expected value with respect to an arbitrary component of $\mathbf{D}_{lm}$, denoted by τ equals

$$\begin{aligned} \frac{\partial E[\tilde{\mathbf{Y}}_{ci} | \tilde{\mathbf{y}}_{bi} = \mathbf{1}]}{\partial \tau} = & \mathbf{E}_i^* \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{E}_i(\mathbf{V}_i^{-1} \mathbf{V}_i^* \mathbf{V}_i^{-1}) \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\ & \mathbf{E}_i^* \mathbf{H}' \mathbf{B}_i^{-1} (\mathbf{F}_i + \mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]) + \\ & \mathbf{E}_i \mathbf{H}_i^* \mathbf{B}_i^{-1} (\mathbf{F}_i + \mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]) - \\ & \mathbf{E}_i \mathbf{H}_i (\mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1}) (\mathbf{F}_i + \mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]) + \\ & \mathbf{E}_i \mathbf{H}_i \mathbf{B}_i^{-1} (t\mathbf{r}^* + \mathbf{F}_i^*) \end{aligned}$$

To allow for a convenient solution for a general case, the following expression was evaluated numerically

$$t\mathbf{r}^* = \frac{\partial \mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]}{\partial \tau}$$

In addition,

$$\begin{aligned}
D_{lm}^* &= \frac{\partial D}{\partial \tau} \\
B_i^* &= B_i \tilde{Z}_{bi} (K_i D^{-1} D_{lm}^* D^{-1} K_i) \tilde{Z}_{bi}' B_i \\
V_i^* &= \tilde{Z}_{ci} D_{lm}^* \tilde{Z}_{ci}' \\
H_i^* &= -B_i^* \tilde{Z}_{bi} K_i \tilde{Z}_{ci}' \Sigma_i^{-1} - B_i \tilde{Z}_{bi}' (K_i D^{-1} D_{lm}^* D^{-1} K_i) \tilde{Z}_{ci} \Sigma_i^{-1} \\
E_i^* &= -E_i \left[-V_i^{-1} V_i^* V_i^{-1} + H_i^{*'} B_i^{-1} H_i + H_i' \left(-\tilde{Z}_{bi} (K_i D^{-1} D_{lm}^* D^{-1} K_i) \tilde{Z}_{bi}' \right) H_i + \right. \\
&\quad \left. H_i' B_i^{-1} H_i^* \right] E_i \\
T_i^* &= -T_i \left[-\tilde{Z}_{bi} (K_i D^{-1} D_{lm}^* D^{-1} K_i) \tilde{Z}_{bi}' - (H_i^{*'} B_i^{-1})' E_i H_i' B_i^{-1} + \right. \\
&\quad (H_i' B_i^{-1} B_i^* B_i^{-1})' E_i H_i' B_i^{-1} - (H_i' B_i^{-1})' E_i^* H_i' B_i^{-1} - (H_i' B_i^{-1})' E_i H_i^* B_i^{-1} - \\
&\quad \left. (H_i' B_i^{-1})' E_i (-H_i' B_i^{-1} B_i^* B_i^{-1}) \right] T_i \\
F_i^* &= T_i^* (H_i' B_i^{-1})' E_i V_i^{-1} \tilde{X}_{ci} \beta + T_i (H_i^{*'} B_i^{-1})' E_i V_i^{-1} \tilde{X}_{ci} \beta + \\
&\quad T_i (-H_i' B_i^{-1} B_i^* B_i^{-1})' E_i V_i^{-1} \tilde{X}_{ci} \beta + T_i (H_i' B_i^{-1})' E_i^* V_i^{-1} \tilde{X}_{ci} \beta + \\
&\quad T_i (H_i' B_i^{-1})' E_i (-V_i^{-1} V_i^* V_i^{-1}) \tilde{X}_{ci} \beta
\end{aligned}$$

Lastly, the derivative of the expected value with respect to σ_c^2 equals

$$\begin{aligned}
\frac{\partial E[\tilde{Y}_{ci} | \tilde{y}_{bi} = 1]}{\partial \sigma_c^2} = & E_i^* V_i^{-1} \tilde{X}_{ci} \beta - E_i (V_i^{-1} S_c^* V_i^{-1}) \tilde{X}_{ci} \beta + \\
& E_i^* H_i' B_i^{-1} (F_i + T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]) + \\
& E_i H_i^{*'} B_i^{-1} (F_i + T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]) - \\
& E_i H_i (B_i^{-1} B_i^* B_i^{-1}) (F_i + T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]) + \\
& E_i H_i B_i^{-1} (F_i^* + tr^*)
\end{aligned}$$

To allow for a convenient solution for a general case, the following expressions was evaluated numerically

$$tr^* = \frac{\partial T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)]}{\partial \sigma_c^2}$$

In addition,

$$\begin{aligned}
\mathbf{S}_c^* &= \frac{\partial \Sigma_i}{\partial \sigma_c^2} \\
\mathbf{K}_i^* &= \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \mathbf{S}_c^* \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} \mathbf{K}_i \\
\mathbf{B}_i^* &= \mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}' \mathbf{B}_i \\
\mathbf{H}_i^* &= -\mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci} \Sigma_i^{-1} - \mathbf{B}_i \tilde{\mathbf{Z}}_{bi}' \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} + \mathbf{B}_i \tilde{\mathbf{Z}}_{bi}' \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \mathbf{S}_c^* \Sigma_i^{-1} \\
\mathbf{E}_i^* &= -\mathbf{E}_i \left[-\mathbf{V}_i^{-1} \mathbf{S}_c^* \mathbf{V}_i^{-1} + \mathbf{H}_i^{*'} \mathbf{B}_i^{-1} \mathbf{H}_i - \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1} \mathbf{H}_i + \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{H}_i^* \right] \mathbf{E}_i \\
\mathbf{T}_i^* &= -\mathbf{T}_i \left[\left(-\mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1} - (\mathbf{H}_i^{*'} \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} - \right. \right. \\
&\quad \left. \left(-\mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1} \right)' \mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} - (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i^* \mathbf{H}_i' \mathbf{B}_i^{-1} - (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{H}_i^{*'} \mathbf{B}_i^{-1} - \right. \\
&\quad \left. \left. (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i (-\mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1}) \right) \right] \mathbf{T}_i \\
\mathbf{F}_i^* &= \mathbf{T}_i^* (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \mathbf{T}_i (\mathbf{H}_i^{*'} \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \mathbf{T}_i (-\mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1})' \mathbf{E}_i \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \mathbf{T}_i (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i^* \mathbf{V}_i^{-1} \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \\
&\quad \mathbf{T}_i (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i (-\mathbf{V}_i^{-1} \mathbf{S}_c^* \mathbf{V}_i^{-1}) \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}
\end{aligned}$$

2.2 Conditional distribution of a subvector of the continuous response given a subvector of the binary responses and a subvector of the continuous responses

The second central moment is calculated in Delporte et al. (2022) and can be simplified using (A.2). This results in the following equation

$$\begin{aligned}
& E \left[(\tilde{\mathbf{Y}}_{ci}^a - E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = 1]) (\tilde{\mathbf{Y}}_{ci}^a - E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = 1])' \right] \\
&= \mathbf{E}_i^{aa} - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} \mathbf{E}_i^{ba} + (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a (\mathbf{N} + \mathbf{J} \mathbf{J}') (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{a'} \\
&\quad + (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a \mathbf{J} ((\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i^{-1} \mathbf{E}_i)^{a'} + (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a \mathbf{J}' (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{a'} + \\
&\quad (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^{a'} \\
&\quad + \left\{ (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a \mathbf{J} \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' - (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a (\mathbf{N} + \mathbf{J} \mathbf{J}') (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} \right. \\
&\quad \left. + (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a \mathbf{J}' (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} \right\} (\mathbf{E}_i^{bb})^{-1} \mathbf{E}_i^{ba} \\
&\quad + \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} \left\{ \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) ((\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i \mathbf{E}_i)^{a'} - (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \mathbf{J} ((\tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})' \mathbf{V}_i \mathbf{E}_i)^a \right. \\
&\quad \left. + \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) \mathbf{J}' (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{a'} - (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b (\mathbf{N} + \mathbf{J} \mathbf{J}') (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{a'} \right\} \\
&\quad + \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} \left\{ (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b (\mathbf{N} + \mathbf{J} \mathbf{J}') (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} \right. \\
&\quad \left. - (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \mathbf{J} \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' \right. \\
&\quad \left. - \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) \mathbf{J}' (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} + \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right) \right. \\
&\quad \left. \left(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b \right)' \right\} (\mathbf{E}_i^{bb})^{-1} \mathbf{E}_i^{ba} - E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = 1] E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = 1]',
\end{aligned}$$

with $\mathbf{J}$ as the expected value of the truncated multivariate normal density, and $\mathbf{N}$ is the second central moment of the latter density. They are defined in B.1.

The standard errors of the transformed parameters are derived by Delporte et al. (2022) with the delta method, but are here simplified with (A.2). The derivative of the expected value with respect to $\boldsymbol{\beta}_{c2}$, an arbitrary coefficient of a predictor of the

continuous response vector X_{c2} is the following:

$$\begin{aligned} \frac{\partial E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi}]}{\partial \beta_{c2}} &= (\mathbf{E}_i \mathbf{V}_i^{-1})^a \tilde{\mathbf{X}}_{c2i} - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1})^b \tilde{\mathbf{X}}_{c2i} \\ &+ \left((\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \right) (\boldsymbol{\nu} + \boldsymbol{\delta}_i), \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{\delta}_i &= \mathbf{T}_i \left(-(\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{c2i})^b + (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i (\mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{c2i}) \right) \\ \boldsymbol{\nu} &= \frac{\sum_{k=1}^{\tilde{p}_i} (\mathbf{H}_i \tilde{\mathbf{X}}_{12i} - \boldsymbol{\delta}_i)_k g_k(o_k) - \boldsymbol{\Theta} \mathbf{T}_i \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix}}{\Phi(\tilde{\mathbf{X}}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}, \mathbf{F}_i, \mathbf{T}_i)}, \\ g_k(\mathbf{x}_k) &= \int_{-\infty}^{o_1} \dots \int_{-\infty}^{o_{i-1}} \int_{-\infty}^{o_{i+1}} \dots \int_{-\infty}^{o_{\tilde{p}_i}} [x_1 \dots x_{k-1} o_k x_{k+1} \dots x_{\tilde{p}_i}]' \phi([x_1 \dots x_{k-1} o_k x_{k+1} \dots x_{\tilde{p}_i}]', \mathbf{T}_i) d\mathbf{x}_{-k} \\ \boldsymbol{\Theta} &= \sum_{k=1}^{\tilde{p}} (\mathbf{H}_{ik} X_{c2i} - \boldsymbol{\delta}_{ik}) \phi(X_{bi} \boldsymbol{\beta} + \mathbf{H}_i X_{ci} \boldsymbol{\beta} - \mathbf{F}_i)_k, \mathbf{T}_{kk} \Phi[(X_{bi} \boldsymbol{\beta} + \mathbf{H}_i X_{ci} \boldsymbol{\beta} - \mathbf{F}_i)_{-k}, \mathbf{T}_{-k|k}], \end{aligned}$$

where $\mathbf{T}_{-k|k}$ is defined in (B.2).

The derivative of the expected value with respect to a coefficient β_{b2} of a predictor $\mathbf{X}_{b2}$ of the binary response vector equals

$$\begin{aligned} \frac{\partial E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = \mathbf{1}]}{\partial \beta_{b2}} &= \left((\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \right) \\ &\frac{\zeta - \Omega \mathbf{T}_i \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix}}{\Phi(\tilde{\mathbf{X}}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}, \mathbf{F}_i, \mathbf{T}_i)}, \end{aligned}$$

with

$$\begin{aligned} \zeta &= \sum_{k=1}^{\tilde{p}_i} \tilde{X}'_{b2ik} g_k(o_k), \\ \Omega &= \sum_{k=1}^{\tilde{p}} X_{b2ik} \phi(\mathbf{X}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \mathbf{X}_{ci} \boldsymbol{\beta} - \mathbf{F}_i)_k, \mathbf{T}_{kk} \Phi[\phi(\mathbf{X}_{bi} \boldsymbol{\beta} + \mathbf{H}_i \mathbf{X}_{ci} \boldsymbol{\beta} - \mathbf{F}_i)_{-k}, \mathbf{T}_{-k|k}]. \end{aligned}$$

The derivative of the expected value with respect to an arbitrary component of $\mathbf{D}_{lm}$,

denoted by τ equals

$$\begin{aligned}
\frac{\partial E[\tilde{\mathbf{Y}}_{ci}^a | \tilde{\mathbf{Y}}_{ci}^b = \tilde{\mathbf{y}}_{ci}^b, \tilde{\mathbf{y}}_{bi} = 1]}{\partial \tau} &= (\mathbf{E}_i^* \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} - \mathbf{E}_i \mathbf{V}_i^{-1} \mathbf{V}_i^* \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^a + \\
&\quad [(\mathbf{E}_i^*)^{ab} (\mathbf{E}_i^{bb})^{-1} - \mathbf{E}_i^{ab} ((\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i^*)^{bb} (\mathbf{E}_i^{bb})^{-1})] \\
&\quad [(\tilde{\mathbf{y}}_{ci}^b - (\mathbf{E}_i \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta})^b)] \\
&+ \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} [-\mathbf{E}_i^* \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta} + \mathbf{E}_i \mathbf{V}_i^{-1} \mathbf{V}_i^* \mathbf{V}_i^{-1} \tilde{\mathbf{X}}_{ci} \boldsymbol{\beta}]^b \\
&+ \left((\mathbf{E}_i^* \mathbf{H}_i' \mathbf{B}_i^{-1} + \mathbf{E}_i \mathbf{H}_i^{*'} \mathbf{B}_i^{-1} - \mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1})^a \right. \\
&\quad \left. - \left[(\mathbf{E}_i^*)^{ab} (\mathbf{E}_i^{bb})^{-1} - \mathbf{E}_i^{ab} ((\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i^*)^{bb} (\mathbf{E}_i^{bb})^{-1}) \right] (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \right. \\
&\quad \left. - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i^* \mathbf{H}_i' \mathbf{B}_i^{-1} + \mathbf{E}_i \mathbf{H}_i^{*'} \mathbf{B}_i^{-1} - \mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1})^b \right) \\
&\times \left(\mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] + \mathbf{F}_i \right) \\
&+ \left((\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^a - \mathbf{E}_i^{ab} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b \right) \mathbf{tr}^*,
\end{aligned}$$

To allow for a convenient solution for a general case, the following expressions was evaluated numerically

$$\mathbf{tr}^* = \frac{\partial \mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] + \mathbf{F}_i}{\partial \tau}$$

In addition,

$$\begin{aligned}
D_{lm}^* &= \frac{\partial D_{lm}}{\partial \tau} \\
B_i^* &= B_i \tilde{Z}_{bi} (K_i D^{-1} D_{lm}^* D^{-1} K_i) \tilde{Z}_{bi}' B_i \\
V_i^* &= \tilde{Z}_{ci} D_{lm}^* \tilde{Z}_{ci}' \\
H_i^* &= -B_i^* \tilde{Z}_{bi} K_i \tilde{Z}_{ci}' \Sigma_i^{-1} - B_i^* \tilde{Z}_{bi}' (K_i D^{-1} D_{lm}^* D^{-1} K_i) \tilde{Z}_{ci} \Sigma_i^{-1} \\
E_i^* &= -E_i \left[-V_i^{-1} V_i^* V_i^{-1} + H_i^{*'} B_i^{-1} H_i + H_i' \left(-\tilde{Z}_{bi}' (K_i D^{-1} D_{lm}^* D^{-1} K_i) \tilde{Z}_{bi} \right) H_i + \right. \\
&\quad \left. H_i' B_i^{-1} H_i^{*'} \right] E_i \\
T_i^* &= -T_i \left[\left(E_i^* H_i' B_i^{-1} + E_i H_i^{*'} B_i^{-1} - E_i H_i (B_i^{-1} B_i^* B_i^{-1}) \right)^{b'} (E_i^{bb})^{-1} (E_i H_i' B_i^{-1})^b - \right. \\
&\quad (E_i H_i' B_i^{-1})^{b'} (E_i^{bb})^{-1} (E_i^*)^{bb} (E_i^{bb})^{-1} (E_i H_i' B_i^{-1})^b + \\
&\quad (E_i H_i' B_i^{-1})^{b'} (E_i^{bb})^{-1} \left(E_i^* H_i' B_i^{-1} + E_i H_i^{*'} B_i^{-1} - E_i H_i (B_i^{-1} B_i^* B_i^{-1}) \right)^b - \\
&\quad (B_i^{-1} B_i^* B_i^{-1}) - (H_i^{*'} B_i^{-1} - H_i' (B_i^{-1} B_i^* B_i^{-1}))' E_i (H_i' B_i^{-1}) - (H_i' B_i^{-1})' E_i^* (H_i' B_i^{-1}) - \\
&\quad \left. (H_i' B_i^{-1})' E_i^* (H_i^{*'} B_i^{-1} - H_i' (B_i^{-1} B_i^* B_i^{-1})) \right] T_i
\end{aligned}$$

Finally, the derivative of the expected value with respect to σ_{ci}^2 , the residual variance of one of the continuous responses, equals

$$\begin{aligned}
\frac{\partial E[\tilde{Y}_{ci}^a | \tilde{Y}_{ci}^b = \tilde{y}_{ci}^b, \tilde{y}_{bi} = 1]}{\partial \sigma_{ci}^2} &= (E_i^* V_i^{-1} \tilde{X}_{ci} \beta - E_i V_i^{-1} S_c^* V_i^{-1} \tilde{X}_{ci} \beta)^a + \\
&\quad [(E_i^*)^{ab} (E_i^{bb})^{-1} - E_i^{ab} ((E_i^{bb})^{-1} (E_i^*)^{bb} (E_i^{bb})^{-1})] \\
&\quad [(\tilde{y}_{ci}^b - (E_i V_i^{-1} \tilde{X}_{ci} \beta)^b)] \\
&\quad + E_i^{ab} (E_i^{bb})^{-1} [-E_i^* V_i^{-1} \tilde{X}_{ci} \beta + E_i V_i^{-1} S_c^* V_i^{-1} \tilde{X}_{ci} \beta]^b \\
&\quad + \left((E_i^* H_i' B_i^{-1} + E_i H_i^{*'} B_i^{-1} - E_i H_i' B_i^{-1} B_i^* B_i^{-1})^a \right. \\
&\quad - [(E_i^*)^{ab} (E_i^{bb})^{-1} - E_i^{ab} ((E_i^{bb})^{-1} (E_i^*)^{bb} (E_i^{bb})^{-1})] (E_i H_i' B_i^{-1})^b \\
&\quad \left. - E_i^{ab} (E_i^{bb})^{-1} (E_i^* H_i' B_i^{-1} + E_i H_i^{*'} B_i^{-1} - E_i H_i' B_i^{-1} B_i^* B_i^{-1})^b \right) \\
&\quad \times \left(T_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] + F_i \right) \\
&\quad + \left((E_i H_i' B_i^{-1})^a - E_i^{ab} (E_i^{bb})^{-1} (E_i H_i' B_i^{-1})^b \right) tr^*,
\end{aligned}$$

To allow for a convenient solution for a general case, the following expression was evaluated numerically

$$\mathbf{tr}^* = \frac{\partial \mathbf{T}_i [-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p)] + \mathbf{F}_i}{\partial \sigma_{c1}^2}$$

In addition,

$$\begin{aligned} \mathbf{S}_c^* &= \frac{\partial \Sigma_i}{\partial \sigma_{c1}^2} \\ \mathbf{K}_i^* &= \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \mathbf{S}_c^* \Sigma_i^{-1} \tilde{\mathbf{Z}}_{ci} \mathbf{K}_i \\ \mathbf{B}_i^* &= \mathbf{B}_i \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i^* \tilde{\mathbf{Z}}_{bi}' \mathbf{B}_i \\ \mathbf{H}_i^* &= -\mathbf{B}_i^* \tilde{\mathbf{Z}}_{bi} \mathbf{K}_i \tilde{\mathbf{Z}}_{ci} \Sigma_i^{-1} - \mathbf{B}_i \tilde{\mathbf{Z}}_{bi}' \mathbf{K}_i^* \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} + \mathbf{B}_i \tilde{\mathbf{Z}}_{bi}' \mathbf{K}_i \tilde{\mathbf{Z}}_{ci}' \Sigma_i^{-1} \mathbf{S}_c^* \Sigma_i^{-1} \\ \mathbf{E}_i^* &= -\mathbf{E}_i \left[-\mathbf{V}_i^{-1} \mathbf{S}_c^* \mathbf{V}_i^{-1} + \mathbf{H}_i^{*'} \mathbf{B}_i^{-1} \mathbf{H}_i - \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1} \mathbf{H}_i + \mathbf{H}_i' \mathbf{B}_i^{-1} \mathbf{H}_i^* \right] \mathbf{E}_i \\ \mathbf{T}_i^* &= -\mathbf{T}_i \left[\left(\mathbf{E}_i^* \mathbf{H}_i' \mathbf{B}_i^{-1} + \mathbf{E}_i \mathbf{H}_i^{*'} \mathbf{B}_i^{-1} - \mathbf{E}_i \mathbf{H}_i (\mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1}) \right)^{b'} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b - \right. \\ &\quad \left. (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i^*)^{bb} (\mathbf{E}_i^{bb})^{-1} (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^b + \right. \\ &\quad \left. (\mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1})^{b'} (\mathbf{E}_i^{bb})^{-1} \left(\mathbf{E}_i^* \mathbf{H}_i' \mathbf{B}_i^{-1} + \mathbf{E}_i \mathbf{H}_i^{*'} \mathbf{B}_i^{-1} - \mathbf{E}_i \mathbf{H}_i (\mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1}) \right)^b - \right. \\ &\quad \left. (\mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1}) - (\mathbf{H}_i^{*'} \mathbf{B}_i^{-1} - \mathbf{H}_i' (\mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1}))' \mathbf{E}_i \mathbf{H}_i' \mathbf{B}_i^{-1} - (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i^* (\mathbf{H}_i' \mathbf{B}_i^{-1}) - \right. \\ &\quad \left. (\mathbf{H}_i' \mathbf{B}_i^{-1})' \mathbf{E}_i^* (\mathbf{H}_i^{*'} \mathbf{B}_i^{-1} \mathbf{H}_i' (\mathbf{B}_i^{-1} \mathbf{B}_i^* \mathbf{B}_i^{-1})) \right] \mathbf{T}_i \end{aligned}$$

C Confidence intervals for conditional probabilities

We will first derive the confidence interval of (2.9). First, the logit transformation is applied to the probability to transform it to the continuous scale:

$$z = \text{logit}\left(f(\tilde{\mathbf{Y}}_{bi} = \mathbf{1} | \tilde{\mathbf{y}}_{ci})\right) = \text{logit}\left(\Phi(\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i)\right).$$

Next the derivative of z with respect to a coefficient β_{c2} of a predictor of the continuous response vector $\mathbf{X}_{c2}$ is derived in order to obtain the transformed standard errors on the continuous scale

$$\frac{\partial z}{\partial \beta_{c2}} = - \frac{\sum_{k=1}^{\tilde{p}_i} \mathbf{H}_{ik} \mathbf{X}_{c2i} \phi[(\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_k; B_{kk}] \Phi[(\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}]}{\left(\Phi[\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]\right)^2 - \Phi[\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]}, \quad (\text{C.1})$$

where kk denotes the element on row k and column k , k denotes the row k of the matrix or element k of the vector. In addition, $\mathbf{B}_i$ is partitioned as follows

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{11}^{(k)} & \mathbf{B}_{12}^{(k)} \\ \mathbf{B}_{21}^{(k)} & B_{kk} \end{bmatrix}.$$

Next, $\mathbf{B}_{-k|k}$ is defined as

$$\mathbf{B}_{-k|k} = \mathbf{B}_{11}^{(k)} - \mathbf{B}_{12}^{(k)} B_{kk}^{-1} \mathbf{B}_{21}^{(k)}, \quad (\text{C.2})$$

which has been retrieved from Poddar (2016), in their Appendix A.

Next, the gradient of a coefficient β_{b2} of one of the predictors of the binary response vector $\mathbf{X}_{b2}$ is defined as

$$\frac{\partial z}{\partial \beta_{b2}} = - \frac{\sum_{k=1}^{\tilde{p}_i} X_{b2ik} \phi[(\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_k; B_{kk}] \Phi[(\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}]}{\left(\Phi[\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]\right)^2 - \Phi[\tilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]}. \quad (\text{C.3})$$

To allow for a convenient solution for a general case, the parts of the two following gradients expressions were evaluated numerically. The first expression shows the gradient of the residual variance σ_{c1}^2 of a continuous response Y_{c1} , and the second ex-

pression contains the gradient of τ , an arbitrary component of the variance-covariance matrix of the random effects $\mathbf{D}$

$$\begin{aligned}\frac{\partial z}{\partial \sigma_{c1}^2} &= \frac{\partial \Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i)}{\partial \sigma_{c1}^2} \frac{-1}{\left(\Phi[\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]\right)^2 - \Phi[\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]}, \\ \frac{\partial z}{\partial \tau} &= \frac{\partial \Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i)}{\partial \tau} \frac{-1}{\left(\Phi[\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]\right)^2 - \Phi[\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i]}.\end{aligned}$$

Hence, the 95% confidence interval can be constructed as

$$\text{expit}\left\{z \pm 1.96\sqrt{\left(\frac{\partial z}{\partial \boldsymbol{\theta}}\right)' \text{Var}(\hat{\boldsymbol{\theta}}) \left(\frac{\partial z}{\partial \boldsymbol{\theta}}\right)}\right\}, \quad (\text{C.4})$$

where $\boldsymbol{\theta}$ signals the vector of estimated parameters.

We will now derive the confidence interval of (2.10). First, let z be a logit transformation of the latter conditional probability. The gradient of a coefficient β_{c2} of a predictor of the continuous responses $\mathbf{X}_{c2}$ is derived:

$$\begin{aligned}\frac{\partial z}{\partial \beta_{c2}} &= \left\{ \sum_{k=1}^{\tilde{p}_i} \mathbf{H}_{ik} \mathbf{X}_{c2i} \phi[(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}] \Phi[(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}] \right. \\ &\times \Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb}) \\ &- \sum_{k=1}^{\tilde{p}_i^b} \mathbf{H}_{ik}^b \mathbf{X}_{c2i} \phi[(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}^{bb}] \Phi[(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}))_{-k}; \mathbf{B}_{-k|k}^{bb}] \\ &\times \left. \Phi[\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i] \right\} \\ &\times \frac{-\left(\Phi[\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb}]\right)^{-2}}{\left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb})}\right)^2 - \left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb})}\right)},\end{aligned} \quad (\text{C.5})$$

where $\mathbf{B}_{-k|k}$ is defined in (C.1).

Next, the gradient of a coefficient β_{b2} of one of the predictors of the binary responses

$\mathbf{X}_{b2}$ is defined as

$$\begin{aligned}
\frac{\partial z}{\partial \beta_{b2}} = & \left\{ \sum_{k=1}^{\tilde{p}_i} X_{b2ik} \phi[(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}] \Phi[(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}] \right. \\
& \times \Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb}) \\
& - \sum_{k=1}^{\tilde{p}_i^b} X_{b2ik} \phi[(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}^{bb}] \Phi[(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}))_{-k}; \mathbf{B}_{-k|k}^{bb}] \\
& \times \left. \Phi[\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i] \right\} \\
& \times \frac{-\left(\Phi[\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb}]\right)^{-2}}{\left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb})}\right)^2 - \left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb})}\right)}.
\end{aligned} \tag{C.6}$$

To allow for a convenient solution for a general case, the following expressions were evaluated numerically

$$\begin{aligned}
st^* &= \frac{\partial \Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i)}{\partial \sigma_{c_1}^2}, \\
sn^* &= \frac{\partial \Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb})}{\partial \sigma_{c_1}^2}, \\
dt^* &= \frac{\partial \Phi(\widetilde{\mathbf{X}}_{bi}\boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i)}{\partial \tau}, \\
dn^* &= \frac{\partial \Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci}\boldsymbol{\beta}); \mathbf{B}_i^{bb})}{\partial \tau}.
\end{aligned}$$

The gradients with respect to the residual error variance $\sigma_{c_1}^2$ and an arbitrary com-

ponent of the variance-covariance matrix of the random effects τ , equal

$$\begin{aligned} \frac{\partial z}{\partial \sigma_{c_1}^2} = & \left\{ st^* \Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb}) - \right. \\ & \left. sn^* \Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i) \right\} \\ & - \left(\Phi[\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb}] \right)^{-2} \\ & \times \frac{\left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)^2 - \left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)}{\left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)^2 - \left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)}, \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} \frac{\partial z}{\partial \tau} = & \left\{ dt^* \Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb}) - \right. \\ & \left. dn^* \Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i) \right\} \\ & - \left(\Phi[\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb}] \right)^{-2} \\ & \times \frac{\left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)^2 - \left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)}{\left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)^2 - \left(\frac{\Phi(\widetilde{\mathbf{X}}_{bi} \boldsymbol{\beta} - \mathbf{H}_i(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i)}{\Phi(\widetilde{\mathbf{X}}_{bi}^b \boldsymbol{\beta} - \mathbf{H}_i^b(\widetilde{\mathbf{Y}}_{ci} - \widetilde{\mathbf{X}}_{ci} \boldsymbol{\beta}); \mathbf{B}_i^{bb})} \right)}. \end{aligned} \quad (\text{C.8})$$

Again, the confidence interval with appropriate bounds can be constructed with (C.4).

D The manifest correlation function

4.1 Proof

The formula of the manifest correlation as described in Delporte et al. (2022) is the following:

$$\rho_{Y_{lij}, Y_{mik}} = \frac{\left(\frac{1}{|\mathbf{D}_{lm}|^{1/2} |\mathbf{M}_i|^{1/2} L_i^{1/2}} - 1 \right) \mathbf{x}'_{lij} \boldsymbol{\beta} \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) + \frac{1}{|\mathbf{D}_{lm}|^{1/2} |\mathbf{M}_i|^{1/2} L_i} \mathbf{z}'_{lij} \mathbf{M}_i^{-1} \mathbf{z}'_{mik} \phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta})}{\sqrt{(\mathbf{z}'_{lij} \mathbf{D}_{lm} \mathbf{z}_{lij} + \Sigma_{lij}) \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) (1 - \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}))}}},$$

where $\mathbf{D}_{lm}$ denotes the submatrix of $\mathbf{D}$ relating to the variances and covariances of the random effects of both responses i and j and

$$\begin{aligned} \mathbf{M}_i &= \mathbf{D}_{lm}^{-1} + \mathbf{z}_{mik} \mathbf{z}'_{mik} \\ L_i &= I - \mathbf{z}'_{mik} \mathbf{M}_i^{-1} \mathbf{z}_{mik}. \end{aligned}$$

Now, consider

$$\begin{aligned} |\mathbf{M}_i| \times |\mathbf{D}| \times L_i &= 1 \\ |\mathbf{M}_i| \times |\mathbf{D}| &= L_i^{-1} \\ |\mathbf{M}_i \times \mathbf{D}| &= (1 - \mathbf{z}'_{mik} \mathbf{M}_i^{-1} \mathbf{z}_{mik})^{-1} \\ |(\mathbf{D}^{-1} + \mathbf{z}_{mik} \mathbf{z}'_{mik}) \times \mathbf{D}| &= 1 - \mathbf{z}'_{mik} (-\mathbf{M}_i + \mathbf{z}_{mik} \mathbf{z}'_{mik})^{-1} \mathbf{z}_{mik} \\ |I + \mathbf{D} \mathbf{z}_{mik} \mathbf{z}'_{mik}| &= 1 - \mathbf{z}'_{mik} (-\mathbf{M}_i + \mathbf{z}_{mik} \mathbf{z}'_{mik})^{-1} \mathbf{z}_{mik} \\ |I + \mathbf{D} \mathbf{z}_{mik} \mathbf{z}'_{mik}| &= 1 - \mathbf{z}'_{mik} (-\mathbf{D}^{-1} - \mathbf{z}_{mik} \mathbf{z}'_{mik} + \mathbf{z}_{mik} \mathbf{z}'_{mik})^{-1} \mathbf{z}_{mik} \\ |I + \mathbf{D} \mathbf{z}_{mik} \mathbf{z}'_{mik}| &= 1 + \mathbf{z}'_{mik} \mathbf{D} \mathbf{z}_{mik} \end{aligned}$$

As a consequence, the expression simplifies to

$$\rho_{Y_{lij}, Y_{mik}} = \frac{\frac{1}{L_i^{1/2}} \mathbf{z}'_{lij} \mathbf{M}_i^{-1} \mathbf{z}_{mik} \phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta})}{\sqrt{(\mathbf{z}'_{lij} \mathbf{D}_{lm} \mathbf{z}_{lij} + \Sigma_{lij}) \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) (1 - \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}))}},$$

4.2 Standard errors

We extended the methodology of Delporte et al. (2022) by calculating the formula of the standard errors of (4.1). The Fisher Z transformation is first applied to (4.1) in order to transform the probability which takes values on the unit interval to a quantity which takes values on the entire real line. Now, the delta method (Oehlert, 1992) has to be applied to calculate the standard errors, since the estimates of the joint model are first entered in formula (4.1) and then transformed with the Fisher z transformation. Hence, the standard error of the Fisher transformed correlation z equals

$$SE(z) = \sqrt{\frac{\partial z}{\partial \boldsymbol{\theta}'} \text{Var}(\hat{\boldsymbol{\theta}}) \frac{\partial z}{\partial \boldsymbol{\theta}}}, \quad (\text{D.1})$$

where $\boldsymbol{\theta}$ indicates the parameter vector.

$\frac{\partial z}{\partial \boldsymbol{\beta}_{m2}}$ for a coefficient of an arbitrary predictor for the binary response $\mathbf{X}_{m2}$ equals

$$\begin{aligned} \frac{\partial z}{\partial \boldsymbol{\beta}_{m2}} = & \frac{-1}{\rho^2 - 1} \frac{1}{\nu^2} \left\{ \nu \frac{-X_{m2ik} \mathbf{x}'_{mik} \boldsymbol{\beta} L_i \phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta})}{L_i^{1/2}} \mathbf{z}'_{lij} \mathbf{M}_i^{-1} \mathbf{z}_{mik} - \right. \\ & \frac{1}{L_i^{1/2}} \mathbf{z}'_{lij} \mathbf{M}_i^{-1} \mathbf{z}_{mik} \phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) \\ & \left. \frac{1}{2\nu} \left[\mathbf{z}'_{lij} \mathbf{D}_{lm} \mathbf{z}_{lij} + \Sigma_{lij} \right] X_{m2ik} \sqrt{L} \phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) (1 - 2\Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta})) \right\} \end{aligned}$$

where ρ equals the non-transformed correlation between Y_{lij} and Y_{mik} and

$$\nu = \sqrt{(\mathbf{z}'_{lij} \mathbf{D}_{lm} \mathbf{z}_{lij} + \Sigma_{lij}) \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) (1 - \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}))}.$$

Next, the derivative of σ_{lij}^2 equals

$$\frac{\partial z}{\partial \sigma_{lij}^2} = \frac{\Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) (1 - \Phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}))}{\rho^2 - 1} \left\{ \frac{1}{L_i^{1/2}} \mathbf{z}'_{lij} \mathbf{M}_i^{-1} \mathbf{z}_{mik} \phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta}) \right\} \frac{1}{2\nu^3}.$$

The derivative of an arbitrary component of $\mathbf{D}_{lm}$, denoted by τ equals

$$\frac{\partial z}{\partial \tau} = \frac{-1}{\rho^2 - 1} \left[\frac{\nu t_1 - t_2 \frac{1}{L_i^{1/2}} \mathbf{z}'_{lij} \mathbf{M}_i^{-1} \mathbf{z}_{mik} \phi(L_i^{1/2} \mathbf{x}'_{mik} \boldsymbol{\beta})}{\nu^2} \right],$$

where

$$\begin{aligned}
\mathbf{D}^* &= \frac{\partial \mathbf{D}_{lm}}{\partial \tau}, \\
\mathbf{M}^* &= -\mathbf{D}_{lm}^{-1} \mathbf{D}^* \mathbf{D}_{lm}^{-1}, \\
L_i^* &= -\mathbf{z}_{mik}' \mathbf{M}_i^{-1} \mathbf{D}_{lm}^{-1} \mathbf{D}^* \mathbf{D}_{lm}^{-1} \mathbf{M}_i^{-1} \mathbf{z}_{mik} \\
\zeta &= \frac{L_i^* \mathbf{x}_{mik}' \boldsymbol{\beta}}{2\sqrt{L_i}} \phi(L_i^{1/2} \mathbf{x}_{mik}' \boldsymbol{\beta}) \\
t_1 &= -\frac{1}{2} L_i^{-3/2} L_i^* \mathbf{z}_{lij}' \mathbf{M}_i^{-1} \mathbf{z}_{mik} \phi(L_i^{1/2} \mathbf{x}_{mik}' \boldsymbol{\beta}) - \frac{1}{L_i^{1/2}} \mathbf{z}_{lik}' \mathbf{M}_i^{-1} \mathbf{M}^* \mathbf{M}_i^{-1} \mathbf{z}_{mik} \phi(L_i^{1/2} \mathbf{x}_{mik}' \boldsymbol{\beta}) - \\
&\quad \frac{1}{2L_i^{1/2}} \mathbf{z}_{lij}' \mathbf{M}_i^{-1} \mathbf{z}_{mik} L_i^* (\mathbf{x}_{mik}' \boldsymbol{\beta})^2 \phi(L_i^{1/2} \mathbf{x}_{mik}' \boldsymbol{\beta}), \\
t_2 &= \frac{1}{2\nu} \left\{ \mathbf{z}_{lij}' \mathbf{D}^* \mathbf{z}_{lij} \Phi(L_i^{1/2} \mathbf{x}_{mik}' \boldsymbol{\beta}) (1 - \Phi(L_i^{1/2} \mathbf{x}_{mik}' \boldsymbol{\beta})) + \right. \\
&\quad \left. \zeta (\mathbf{z}_{lij}' \mathbf{D}_{lm} \mathbf{z}_{lij} + \Sigma_{lij}) (1 - 2\Phi(L_i^{1/2} \mathbf{x}_{mik}' \boldsymbol{\beta})) \right\}.
\end{aligned}$$

E Pairwise modelling of independent subsamples

The computational complexity of our models in the case studies is reduced by the use of multiple pseudo-likelihood methods. Firstly, the pairwise method of Fieuws and Verbeke (2006) is implemented. This method estimates the parameters in the joint model by fitting a bivariate model for each pair of the responses. As a result, some parameters are estimated only once (e.g., the covariances between the random effects of different responses), while other parameters are estimated multiple times (e.g., the variance of the random intercept of a response). To obtain a single estimate for the parameters that are estimated multiple times, an appropriately weighted average of the estimated values can be used. More details and the calculation of the standard errors can be found in Fieuws and Verbeke (2006).

Secondly, the partitioned samples method of Molenberghs et al. (2011) can be of use to reduce computational complexity. In this method, the dataset is divided into random subsamples and each sample is analysed separately. This results in multiple estimates for each parameter, which can be transformed into a single estimate by calculating an appropriate weighted average.

Thirdly, the latter two methods can be combined to reduce the computational complexity even more. Hence, we divide the data in subsamples and proceed to analyse each sample with the pairwise method. As a result, we have estimates for each pair of responses for each subsample. Again, we can combine those estimates by calculating the appropriately weighted average to obtain a single estimate for each parameter. More details can be found in Ivanova et al. (2017).

F Case study: Covid-19

Table 10: Parameter estimates (standard errors) of perceived infectability and germ aversion.

Effect	Infectability		Germ aversion	
Intercept	4.536	(0.199)	4.428	(0.026)
Time 1	-0.026	(0.074)	-0.022	(0.007)
Time 2	0.019	(0.083)	0.068	(0.009)
Time 3	-0.145	(0.095)	0.006	(0.011)
Time 4	-0.126	(0.111)	-0.198	(0.015)
Gender: Male	-0.225	(0.069)	-0.343	(0.004)
Large city	0.135	(0.111)	0.072	(0.008)
Suburbs	-0.006	(0.089)	-0.054	(0.006)
Small city	0.112	(0.079)	0.096	(0.005)
Countryside	-0.109	(0.153)	-0.115	(0.015)
Age	-0.002	(0.003)	0.015	(<0.001)
Student	-0.100	(0.146)	0.01	(0.016)
Permanent disability	0.498	(0.211)	0.451	(0.029)
Children: No	0.080	(0.075)	0.121	(0.004)
No parents <60 alive	0.154	(0.077)	0.125	(0.004)
Perceived income	-0.177	(0.029)	-0.121	(0.001)
Time 1 x Gender: Male	0.001	(0.045)	0.072	(0.002)
Time 2 x Gender: Male	-0.020	(0.048)	-0.123	(0.003)
Time 3 x Gender: Male	0.025	(0.057)	-0.089	(0.004)
Time 4 x Gender: Male	-0.014	(0.061)	-0.071	(0.004)
Time 1 x Age	0.000	(0.001)	0.004	(<0.001)
Time 2 x Age	-0.004	(0.002)	0.003	(<0.001)
Time 3 x Age	-0.001	(0.002)	0.003	(<0.001)
Time 4 x Age	-0.001	(0.002)	0.005	(<0.001)

Table 11: Parameter estimates (standard errors) of quality newspaper, social media, internet.

Effect	Quality paper		Social media		Internet	
Intercept	-1.805	(0.476)	0.445	(0.308)	0.505	(0.282)
Time 1	-0.624	(0.313)	-0.658	(0.262)	-0.149	(0.245)
Time 2	-0.682	(0.337)	-0.914	(0.294)	-0.521	(0.262)
Time 3	-0.467	(0.38)	-0.84	(0.339)	-0.632	(0.300)
Time 4	-0.56	(0.456)	-1.257	(0.391)	-0.901	(0.308)
Gender: Male	-0.158	(0.178)	-0.04	(0.122)	0.084	(0.115)
Large city	0.871	(0.259)	0.523	(0.180)	0.211	(0.157)
Suburbs	0.285	(0.218)	0.184	(0.136)	0.108	(0.124)
Small city	0.127	(0.200)	0.135	(0.127)	0.066	(0.112)
Countryside	-0.047	(0.339)	-0.239	(0.192)	0.089	(0.167)
Age	0.005	(0.007)	-0.012	(0.005)	-0.002	(0.005)
Student	0.88	(0.360)	0.259	(0.241)	-0.165	(0.224)
Permanent disability	-0.574	(0.496)	-0.047	(0.280)	-0.097	(0.261)
Children: No	0.162	(0.171)	-0.031	(0.112)	0.147	(0.100)
No parents <60 alive	0.216	(0.205)	0.021	(0.126)	0.010	(0.112)
Perceived income	0.18	(0.068)	0.086	(0.044)	-0.053	(0.038)
Time 1 x Gender: Male	0.138	(0.177)	0.024	(0.145)	-0.045	(0.141)
Time 2 x Gender: Male	0.51	(0.187)	0.225	(0.158)	0.016	(0.147)
Time 3 x Gender: Male	0.731	(0.216)	0.255	(0.187)	0.156	(0.167)
Time 4 x Gender: Male	0.705	(0.245)	0.269	(0.216)	0.134	(0.178)
Time 1 x Age	0.008	(0.006)	0.007	(0.005)	-0.001	(0.005)
Time 2 x Age	0.001	(0.006)	0.007	(0.005)	0.002	(0.005)
Time 3 x Age	-0.004	(0.007)	0.003	(0.006)	0.005	(0.006)
Time 4 x Age	-0.002	(0.008)	0.007	(0.007)	0.007	(0.006)

Table 12: Manifest correlations between perceived infectability and usage of social Media of quality newspapers.

	Wave(SNS)				
Wave(infect)	0	1	2	3	4
0	.041[.040;.042]	.033[.032;.034]	.028[.027;.029]	.024[.022;.025]	.018[.016;.020]
1	.036[.035;.037]	.029[.028;.030]	.024[.023;.025]	.020[.019;.021]	.015[.013;.017]
2	.031[.030;.032]	.024[.023;.025]	.020[.019;.021]	.016[.015;.018]	.012[.010;.014]
3	.025[.024;.026]	.020[.019;.021]	.016[.015;.017]	.013[.011;.014]	.009[.007;.011]
4	.020[.019;.021]	.015[.014;.016]	.012[.011;.013]	.009[.008;.011]	.006[.004;.008]

Table 13: Manifest correlations between perceived infectability and internet usage .

	Wave(internet)				
Wave(infect)	0	1	2	3	4
0	.037[.036;.038]	.040[.039;.041]	.043[.042;.044]	.050[.049;.052]	.053[.051;.055]
1	.032[.031;.033]	.036[.035;.037]	.041[.040;.042]	.049[.048;.050]	.053[.051;.054]
2	.026[.025;.028]	.032[.031;.033]	.038[.037;.039]	.047[.046;.048]	.052[.050;.054]
3	.021[.020;.022]	.028[.027;.029]	.035[.034;.036]	.045[.043;.046]	.051[.049;.053]
4	.015[.014;.017]	.024[.022;.025]	.032[.030;.033]	.042[.041;.044]	.049[.047;.051]

Table 14: Manifest correlations between germ aversion and quality newspaper usage.

	Wave(paper)				
Wave(germ)	0	1	2	3	4
0	-.009[.010;-.008]	.004[.003;.005]	.015[.014;.016]	.025[.024;.027]	.035[.033;.037]
1	-.010[.011;-.009]	.002[.001;.002]	.012[.011;.013]	.021[.020;.023]	.030[.028;.032]
2	-.011[.012;-.010]	-.001[.002;.000]	.009[.008;.010]	.017[.016;.018]	.025[.023;.026]
3	-.012[.013;-.011]	-.003[.004;-.002]	.005[.004;.007]	.013[.011;.014]	.019[.018;.021]
4	-.013[.014;-.011]	-.005[.006;-.004]	.002[.001;.004]	.008[.007;.010]	.014[.012;.016]

Table 15: Manifest correlations between germ aversion and usage of social media of quality newspapers.

	Wave(SNS)				
Wave(germ)	0	1	2	3	4
0	.023[.022;.024]	.019[.018;.020]	.017[.016;.018]	.015[.014;.017]	.012[.010;.015]
1	.012[.011;.013]	.011[.010;.012]	.011[.010;.012]	.011[.010;.013]	.010[.008;.012]
2	.001[.000;.002]	.003[.002;.004]	.005[.004;.006]	.007[.006;.009]	.008[.006;.010]
3	-.010[-.011;-.009]	-.005[-.006;-.004]	.000[-.001;.001]	.003[.002;.005]	.006[.004;.008]
4	-.020[-.021;-.019]	-.012[-.013;-.011]	-.006[-.007;-.004]	.000[-.002;.001]	.004[.002;.006]

Table 16: Manifest correlations between germ aversion and internet usage

	Wave(internet)				
Wave(germ)	0	1	2	3	4
0	.037[.036;.038]	.030[.029;.031]	.024[.023;.025]	.021[.019;.022]	.015[.013;.017]
1	.032[.031;.033]	.028[.027;.029]	.026[.025;.027]	.025[.024;.026]	.022[.020;.024]
2	.027[.026;.028]	.026[.025;.027]	.026[.026;.027]	.029[.028;.030]	.029[.027;.030]
3	.021[.020;.022]	.024[.023;.025]	.027[.026;.028]	.032[.031;.034]	.034[.032;.036]
4	.016[.015;.017]	.022[.020;.023]	.027[.026;.028]	.035[.034;.037]	.040[.038;.042]

G Vaccination data

7.1 SAS Code

A sample of the data and the code of this section can also be found in the Supplementary materials.

Obs	SurveyID	wave	bereid_of_vac	bereid_of_vac_a	CO_Outcome_Pos	CO_Infection	gender	age_group	region
1	1.00	1	0	0	12.50	7.50	2	2	Flanders
2	1.00	2	1	1	-1.00	-3.50	2	2	Flanders
3	1.00	3	1	1	26.50	-13.50	2	2	Flanders
4	1.00	4	1	1	-7.50	-12.50	2	2	Flanders
5	1.00	5	1	1	-4.50	4.00	2	2	Flanders
6	2.00	1	1	1	8.00	31.00	2	2	Flanders
7	2.00	2	.	.	.	.	2	2	Flanders
8	2.00	3	.	.	.	.	2	2	Flanders
9	2.00	4	1	1	3.00	7.00	2	2	Flanders
10	2.00	5	1	1	-5.00	-10.00	2	2	Flanders

Figure 6: First ten observations in the dataset

Figure 6 shows the first ten observations in the dataset. In order to analyze the data, each measurement of each response of each individual had to be on a separate record. This was done using the following SAS code

```

data v.final2;
length distvar $11;
length response $11;
length linkvar $11;
length var $11;
set v.final2;
time=wave-1;
response = bereid_of_vac;
var='bereid';
distvar   = "Binary";
linkvar   = "PROBIT";
output;
response = bereid_of_vac_a;

```



```

var='bereidA';
distvar    = "Binary";
linkvar    = "PROBIT";
output;
response = co_infection;
var='CI';
distvar    = "Normal";
linkvar    = "IDEN";
output;
response = CO_Outcome_Neg;
var='CON';
distvar    = "Normal";
linkvar    = "IDEN";
output;
run;

```

In addition, we created separate variables to use later for the random intercept and randoms slope for each variable. Since the variances of the random effects were expected to be small, the values were divided by ten. In addition, we created separate time indicators for the continuous responses.

```

data v.final2;
set v.final2;
if var= "CI" then do;
ci_int=1;
ci_t=time/10;
end;
else do;
ci_int=0;
ci_t=0;
end;
if var= "bereidA" then do;
bereidA_int=1;
bereidA_t=time/10;
end;
else do;
bereidA_int=0;
bereidA_t=0;
end;

```

```

end;
if var= "bereid" then do;
  bereid_int=1;
  bereid_t=time/10;
end;
else do;
  bereid_int=0;
  bereid_t=0;
end;
if var= "CON" then do;
  con_int=1;
  con_t=time/10;
end;
else do;
  con_int=0;
  con_t=0;
end;
run;

data v.final2;
set v.final2;
CITIMECLSS=ci_t*10;
contimeclss=con_t*10;
run;

```

We then took a random sample, and analyzed each pair of responses separately on each sample. For the bivariate analysis of a pair of binary responses or a pair of mixed response types, the analysis was performed in SAS PROC GLIMMIX. The following code provides an example

```

proc glimmix data=vac2.s5 method=quad(qpoints=5) initglm asycov GRADIENT
  HESSIAN SUBGRADIENT=vac2.g_cib5;
class var citimeclss(ref='1') surveyid distvar linkvar age_group(ref='4')
  region(ref='Flanders') ;
nloptions maxfunc=10000 maxiter=10000 technique=newrap;
model response(ref='1') = var
  citimeclss bereid_int*wave_1 bereid_int*time
  var*gender_1 var*age_group var*region

```

```

gender_1*citimeclss age_group*citimeclss region*citimeclss
gender_1*bereid_int*wave_1 age_group*bereid_int*wave_1
    region*bereid_int*wave_1
gender_1*bereid_int*time age_group*bereid_int*time region*bereid_int*time
/noint s dist=byobs(distvar) link=byobs(linkvar) solution;
random ci_int ci_t bereid_int bereid_t/type=un subject=surveyID ;
ods output hessian=vac2.h_cib5 parameterestimates=vac2.parms_cib5
    CovParms=vac2.r_cib5;
where ci_int=1 or bereid_int=1;
run;

```

For a pair of continuous responses, the starting values for the parameter estimates were obtained via univariate linear mixed models in SAS PROC MIXED. Next, the responses were jointly analyzed via SAS PROC NLMIXED

```

proc nlmixed data=v.s5 qpoints=5 maxiter=10000
maxfunc=100000 technique=newrap hess subgrad=v.g_cicon5;
/*starting values for the parameter estimates*/
parms
/*CON*/
beta201 = 1.0742
beta202 = -1.6482
beta203 = -2.2175
beta204 = 0.01856
beta205 = 0.9816
beta206 = 3.1154
beta207 = -4.7174
beta208 = -1.5571
beta209 = -3.7211
beta210 = -5.3718
beta211 = 2.7919
beta212 = 3.298
beta213 = 1.0026
beta214 = -2.2369
beta215 = -4.1239
beta216 = -3.7082
beta217 = -3.4485
beta218 = 4.852

```

```

beta219 = 0.4306
beta220 = 5.381
beta221 = 8.4258
beta222 = 1.4438
beta223 = 7.1814
beta224 = -0.5756
beta225 = 6.4732
beta226 = 6.1074
beta227 = 3.3694
beta228 = 8.8807
beta229 = -0.3548
beta230 = 2.0902
beta231 = 6.9178
beta232 = 1.0865
beta233 = 2.7396
beta234 = -0.2869
beta235 = 2.2958
beta236 = 3.8862
beta237 = -2.4378
beta238 = 0.8178
beta239 = -0.4347
beta240 = -1.4864
beta241 = 3.6942
beta242 = 1.3872
beta243 = 0.664
beta244 = -0.2542
beta245 = 2.2136

```

```
/*ci*/
```

```

beta11 = 9.0346
beta12 = -1.0924
beta13 = -2.6435
beta14 = 3.9815
beta15 = -0.09557
beta16 = 0.02026
beta21 = 12.6359
beta22 = 2.264
beta23 = 0.2143

```

```
beta24 = 0.6565
beta25 = 4.5365
beta26 = 1.35
beta27 = -0.2102
beta32 = 1.4685
beta33 = 0.8375
beta34 = -2.3893
beta35 = -1.7486
beta36 = -9.9476
beta37 = -4.1605
beta38 = 2.7575
beta39 = 2.1868
beta40 = -2.1077
beta41 = -9.0811
beta42 = -5.3943
beta43 = 1.2656
beta44 = -0.3119
beta45 = 1.5218
beta46 = -7.7712
beta47 = -6.8263
beta48 = -1.4169
beta49 = -3.7714
beta50 = -1.7791
beta51 = -5.0436
beta52 = -1.9424
beta53 = 0.4107
beta54 = -0.5557
beta55 = -5.3334
beta56 = -0.1553
beta57 = 0.6125
beta58 = 2.0198
beta59 = 1.2917
beta60 = -0.9959
beta61 = 0.4817
beta62 = 1.5869
beta63 = 4.8991
```

```
/*initial random effects estimates*/
```

```
ri_d=141.39
ris_d=-103.56
rs_d=465.55
rii_ds=41.2709
rsi_ds=-51.4533
ri_s=153.86
ris_ds=-108.01
rss_ds=510.96
ris_s=-115.37
rs_s=777.01
res_d=211.42
res_s=166.11;
```

```
if var='CI' then do;
```

```
mean= u1+ u2*ci_t + beta11
+ beta12 * wave_2
+ beta13 * wave_3
+ beta14 * wave_4
+ beta15 * wave_5
+ beta16 * gender_1
+ beta21 * age_group_1
+ beta22 * age_group_2
+ beta23 * age_group_3
+ beta24 * age_group_5
+ beta25 * age_group_6
+ beta26 * region_brussels
+ beta27 * region_wallonia
+ beta32 * gender_1*wave_2
+ beta33 * gender_1*wave_3
+ beta34 * gender_1*wave_4
+ beta35 * gender_1*wave_5
+ beta36 * age_group_1*wave_2
+ beta37 * age_group_2*wave_2
+ beta38 * age_group_3*wave_2
+ beta39 * age_group_5*wave_2
```

```

+ beta40 * age_group_6*wave_2
+ beta41 * age_group_1*wave_3
+ beta42 * age_group_2*wave_3
+ beta43 * age_group_3*wave_3
+ beta44 * age_group_5*wave_3
+ beta45 * age_group_6*wave_3
+ beta46 * age_group_1*wave_4
+ beta47 * age_group_2*wave_4
+ beta48 * age_group_3*wave_4
+ beta49 * age_group_5*wave_4
+ beta50 * age_group_6*wave_4
+ beta51 * age_group_1*wave_5
+ beta52 * age_group_2*wave_5
+ beta53 * age_group_3*wave_5
+ beta54 * age_group_5*wave_5
+ beta55 * age_group_6*wave_5
+ beta56 * region_brussels*wave_2
+ beta57 * region_wallonia*wave_2
+ beta58 * region_brussels*wave_3
+ beta59 * region_wallonia*wave_3
+ beta60 * region_brussels*wave_4
+ beta61 * region_wallonia*wave_4
+ beta62 * region_brussels*wave_5
+ beta63 * region_wallonia*wave_5;
dens1 = -0.5*log(3.14159265358) - log(sqrt(res_d))
-0.5*(response-mean)**2/(res_d);
ll = dens1;
end;

if var='CON' then do;
mean= u5+ u6*con_t + beta201
+ beta202 * wave_2
+ beta203 * wave_3
+ beta204 * wave_4
+ beta205 * wave_5
+ beta206 * gender_1
+ beta207 * age_group_1
+ beta208 * age_group_2

```

```

+ beta209 * age_group_3
+ beta210 * age_group_5
+ beta211 * age_group_6
+ beta212 * region_brussels
+ beta213 * region_wallonia
+ beta214 * gender_1*wave_2
+ beta215 * gender_1*wave_3
+ beta216 * gender_1*wave_4
+ beta217 * gender_1*wave_5
+ beta218 * age_group_1*wave_2
+ beta219 * age_group_2*wave_2
+ beta220 * age_group_3*wave_2
+ beta221 * age_group_5*wave_2
+ beta222 * age_group_6*wave_2
+ beta223 * age_group_1*wave_3
+ beta224 * age_group_2*wave_3
+ beta225 * age_group_3*wave_3
+ beta226 * age_group_5*wave_3
+ beta227 * age_group_6*wave_3
+ beta228 * age_group_1*wave_4
+ beta229 * age_group_2*wave_4
+ beta230 * age_group_3*wave_4
+ beta231 * age_group_5*wave_4
+ beta232 * age_group_6*wave_4
+ beta233 * age_group_1*wave_5
+ beta234 * age_group_2*wave_5
+ beta235 * age_group_3*wave_5
+ beta236 * age_group_5*wave_5
+ beta237 * age_group_6*wave_5
+ beta238 * region_brussels*wave_2
+ beta239 * region_wallonia*wave_2
+ beta240 * region_brussels*wave_3
+ beta241 * region_wallonia*wave_3
+ beta242 * region_brussels*wave_4
+ beta243 * region_wallonia*wave_4
+ beta244 * region_brussels*wave_5
+ beta245 * region_wallonia*wave_5;
dens = -0.5*log(3.14159265358) - log(sqrt(res_s))

```



```

-0.5*(response-mean)**2/(res_s);
ll = dens;
end;

model response~general(ll);
random u1 u2 u5 u6 ~normal([0,0,0,0],[ri_d, ris_d ,rs_d, rii_ds, rsi_ds,
    ri_s, ris_ds, rss_ds, ris_s, rs_s]) subject=surveyid;
ods output hessian=v.h_cicon5 parameterestimates=v.parms_cicon5;
where var='CON' or var='CI';
run;

```

7.2 Tables

Table 18: *Parameter estimates (standard errors) of comparative optimism of infection and comparative optimism of severe outcomes.*

Effect	Infection		Severe outcomes	
Intercept	10.409	(1.773)	-0.250	(1.558)
January	-0.319	(2.234)	-0.213	(1.925)
February	-1.619	(2.120)	-0.886	(1.872)
March	0.928	(2.270)	0.832	(2.043)
May	0.591	(2.261)	0.564	(2.078)
Male	-1.130	(1.549)	0.422	(1.303)
I(Age $\leq$ 24)	2.679	(3.538)	0.422	(1.303)
I(25 $\leq$ Age $\leq$ 34)	0.616	(2.382)	1.558	(1.888)
I(35 $\leq$ Age $\leq$ 44)	0.147	(2.396)	-0.637	(2.037)
I(55 $\leq$ Age $\leq$ 64)	-0.176	(2.412)	-0.284	(2.127)
I(65 $\leq$ Age)	2.294	(2.301)	3.741	(2.064)
Brussels capital region	-1.402	(3.001)	0.550	(2.265)
Walloon region	1.006	(1.759)	1.131	(1.417)
January $\times$ Male	0.527	(1.941)	-0.294	(1.614)

February×Male	0.385	(1.906)	-0.059	(1.619)
March×Male	0.458	(1.947)	-1.089	(1.692)
May×Male	0.340	(1.998)	-0.258	(1.766)
January×I(Age≤ 24)	-0.931	(4.539)	-2.316	(3.596)
January×I(25 ≤Age≤ 34)	-1.842	(2.986)	-0.903	(2.440)
January×I(35 ≤Age≤ 44)	0.497	(3.007)	1.641	(2.477)
January×I(55 ≤Age≤ 64)	0.895	(2.975)	0.827	(2.569)
January×I(65 ≤Age)	-0.786	(2.861)	-0.075	(2.461)
February×I(Age≤ 24)	-0.941	(4.795)	1.220	(3.923)
February×I(25 ≤Age≤ 34)	-0.754	(2.979)	-0.930	(2.487)
February×I(35 ≤Age≤ 44)	-0.235	(2.940)	1.879	(2.486)
February×I(55 ≤Age≤ 64)	1.739	(2.869)	1.607	(2.566)
February×I(65 ≤Age)	-0.757	(2.793)	0.002	(2.417)
March×I(Age≤ 24)	-2.209	(4.912)	-0.191	(3.597)
March×I(25 ≤Age≤ 34)	-2.911	(3.244)	0.053	(2.619)
March×I(35 ≤Age≤ 44)	-1.410	(2.960)	0.597	(2.610)
March×I(55 ≤Age≤ 64)	-0.390	(2.982)	0.083	(2.720)
March×I(65 ≤Age)	-2.396	(2.834)	-1.769	(2.624)
May×I(Age≤ 24)	-1.098	(4.999)	0.389	(3.978)
May×I(25 ≤Age≤ 34)	-1.529	(3.207)	-0.323	(2.638)
May×I(35 ≤Age≤ 44)	-1.093	(3.024)	0.378	(2.736)
May×I(55 ≤Age≤ 64)	0.408	(3.009)	0.292	(2.859)
May×I(55 ≤Age≤ 64)	-4.566	(2.902)	-2.258	(2.646)
January×Brussels capital region	0.301	(3.535)	1.497	(2.876)
January×Walloon region	-1.682	(2.262)	0.339	(1.812)
February×Brussels capital region	0.581	(3.410)	1.872	(2.727)
February×Walloon region	-0.235	(2.240)	0.225	(1.840)
March×Brussels capital region	-1.105	(3.697)	2.662	(2.919)

March×Walloon region	-2.210	(2.249)	0.594	(1.890)
May×Brussels capital region	0.876	(3.780)	2.734	(3.098)
May×Walloon region	-0.160	(2.320)	1.020	(1.967)

Table 17: Parameter estimates(standard errors) of own vaccination hesitancy and perceived vaccination hesitancy of peers.

Effect	Own vaccination hesitancy		Vaccination hesitancy peers	
Intercept	2.427	(0.461)	0.150	(0.252)
December	-1.946	(0.489)	0.366	(0.311)
Time	-3.981	(0.503)	-1.227	(0.152)
Male	0.042	(0.370)	-0.073	(0.218)
I(Age $\leq$ 24)	-0.238	(0.801)	0.198	(0.446)
I(25 $\leq$ Age $\leq$ 34)	-0.576	(0.590)	0.140	(0.318)
I(35 $\leq$ Age $\leq$ 44)	-0.035	(0.553)	0.123	(0.334)
I(55 $\leq$ Age $\leq$ 64)	0.528	(0.625)	-0.021	(0.361)
I(65 $\leq$ Age)	0.957	(0.643)	-0.377	(0.366)
Brussels capital region	-1.011	(0.599)	0.329	(0.368)
Walloon region	-0.801	(0.423)	0.319	(0.233)
December $\times$ Male	0.498	(0.413)	0.309	(0.265)
Time $\times$ Male	-0.419	(0.346)	-0.098	(0.129)
December $\times$ I(Age $\leq$ 24)	-0.244	(0.893)	-0.567	(0.530)
December $\times$ I(25 $\leq$ Age $\leq$ 34)	0.157	(0.644)	-0.533	(0.395)
December $\times$ I(35 $\leq$ Age $\leq$ 44)	-0.207	(0.610)	-0.343	(0.409)
December $\times$ I(55 $\leq$ Age $\leq$ 64)	-0.088	(0.669)	0.245	(0.444)
December $\times$ I(65 $\leq$ Age)	0.044	(0.708)	0.842	(0.435)
Time $\times$ I(Age $\leq$ 24)	0.951	(0.698)	0.368	(0.258)
Time $\times$ I(25 $\leq$ Age $\leq$ 34)	1.193	(0.567)	0.367	(0.189)
Time $\times$ I(35 $\leq$ Age $\leq$ 44)	0.469	(0.548)	0.185	(0.193)
Time $\times$ I(55 $\leq$ Age $\leq$ 64)	-1.110	(0.672)	-0.324	(0.226)
Time $\times$ I(65 $\leq$ Age)	-2.303	(0.639)	-0.401	(0.221)
December $\times$ Brussels capital region	0.335	(0.678)	-0.628	(0.450)
December $\times$ Walloon region	-0.184	(0.459)	-0.871	(0.285)
Time $\times$ Brussels capital region	1.886	(0.528)	0.468	(0.209)
Time $\times$ Walloon region	2.052	(0.426)	0.579	(0.139)

7.3 Comparative optimism and vaccination intention peers

Table 19: Manifest correlations between vaccine intention peers and comparative optimism of infection.

Wave(opticism)	Wave(Intention)				
	Intention 1	Intention 2	Intention 3	Intention 4	Intention 5
1	.016[-.011;.043]	-.006[-.032;.019]	-.017[-.044;.011]	-.019[-.048;.009]	-.021[-.049;.008]
2	.015[-.011;.040]	-.005[-.029;.019]	-.014[-.039;.011]	-.017[-.042;.009]	-.018[-.044;.008]
3	.013[-.012;.038]	-.004[-.027;.020]	-.012[-.035;.012]	-.014[-.038;.010]	-.015[-.039;.009]
4	.011[-.014;.036]	-.002[-.026;.021]	-.009[-.033;.015]	-.011[-.035;.014]	-.012[-.036;.013]
5	.009[-.017;.035]	-.001[-.026;.024]	-.006[-.032;.020]	-.007[-.034;.019]	-.008[-.035;.019]

Table 20: Manifest correlations between vaccine intention peers and comparative optimism of severe outcomes.

Wave(opticism)	Wave(Intention)				
	Intention 1	Intention 2	Intention 3	Intention 4	Intention 5
1	.011[-.015;.038]	-.002[-.028;.024]	-.009[-.037;.018]	-.011[-.039;.017]	-.012[-.041;.016]
2	.011[-.015;.036]	.003[-.021;.027]	-.003[-.028;.022]	-.005[-.030;.021]	-.006[-.031;.020]
3	.010[-.016;.035]	.008[-.016;.032]	.004[-.020;.028]	.002[-.022;.026]	.001[-.023;.026]
4	.008[-.018;.034]	.013[-.011;.037]	.010[-.014;.035]	.009[-.016;.034]	.008[-.018;.033]
5	.007[-.020;.034]	.017[-.008;.043]	.016[-.011;.042]	.014[-.013;.041]	.014[-.014;.041]