APPENDIX

A Joint Normal-Ordinal(Probit) Model for Ordinal and Continuous Longitudinal Data

Supplementary Materials

A. Computation of the marginal ordinal random-effects model

We will derive the marginal density of the ordinal random-effects model (3.4).

$$
\begin{aligned}
&f(\boldsymbol{y}_i \leqslant \boldsymbol{c}) \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \frac{1}{(2\pi)^{q/2}|\boldsymbol{D}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{b}_i'\boldsymbol{D}^{-1}\boldsymbol{b}_i\big]\Big\}\Phi(\boldsymbol{\gamma_c} - \boldsymbol{X}_i\boldsymbol{\beta} - \boldsymbol{Z}_i\boldsymbol{b}_i)d\boldsymbol{b}_i \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{t=-\infty}^{t=\boldsymbol{\gamma_c}-\boldsymbol{X}_i\boldsymbol{\beta}-\boldsymbol{Z}_i\boldsymbol{b}_i} \frac{1}{(2\pi)^{(q+p_i)/2}|\boldsymbol{D}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{b}_i'\boldsymbol{D}^{-1}\boldsymbol{b}_i + \boldsymbol{t}'\boldsymbol{t}\big]\Big\}d\boldsymbol{b}_i d\boldsymbol{t} \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{s=-\infty}^{s=\boldsymbol{\gamma_c}-\boldsymbol{X}_i\boldsymbol{\beta}} \frac{1}{(2\pi)^{(q+p_i)/2}|\boldsymbol{D}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{b}_i'\boldsymbol{D}^{-1}\boldsymbol{b}_i + (\boldsymbol{s}-\boldsymbol{Z}_i\boldsymbol{b}_i)'(\boldsymbol{s}-\boldsymbol{Z}_i\boldsymbol{b}_i)\big]\Big\}d\boldsymbol{b}_i d\boldsymbol{s} \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{s=-\infty}^{s=\boldsymbol{\gamma_c}-\boldsymbol{X}_i\boldsymbol{\beta}} \frac{1}{(2\pi)^{(q+p_i)/2}|\boldsymbol{D}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[(\boldsymbol{b}_i-\boldsymbol{\alpha}_i)'\boldsymbol{K}_i(\boldsymbol{b}_i-\boldsymbol{\alpha}_i) + \zeta_i\big]\Big\}d\boldsymbol{b}_i d\boldsymbol{s},
\end{aligned}
$$

Here, $\boldsymbol{\gamma_c}$ denotes the vector of thresholds for category c. In addition, the variable t is rewritten as $\boldsymbol{t} = \boldsymbol{s} - \boldsymbol{Z}_i\boldsymbol{b}_i$ Next,

$$\boldsymbol{b}_i'\boldsymbol{D}^{-1}\boldsymbol{b}_i + (\boldsymbol{s}-\boldsymbol{Z}_i\boldsymbol{b}_i)'(\boldsymbol{s}-\boldsymbol{Z}_i\boldsymbol{b}_i) = (\boldsymbol{b}_i-\boldsymbol{\alpha_i})'\boldsymbol{K}_i(\boldsymbol{b}_i-\boldsymbol{\alpha_i}) + \zeta_i,$$

where

$$\boldsymbol{K}_i = \boldsymbol{D}^{-1} + \boldsymbol{Z}_i'\boldsymbol{Z}_i,$$

$$\boldsymbol{\alpha}_i = \boldsymbol{K}_i^{-1}\boldsymbol{Z}_i's,$$

$$\zeta_i = -\boldsymbol{\alpha}_i'\boldsymbol{K}_i\boldsymbol{\alpha}_i + \boldsymbol{s}'\boldsymbol{s}.$$

Integrating over $\mathbf{b}_i$ results in the following:

$$\begin{aligned}
&f(\boldsymbol{y}_i \leqslant \boldsymbol{c}) \\
&= \int_{s=-\infty}^{s=\boldsymbol{\gamma_c}-\boldsymbol{X}_i\boldsymbol{\beta}} \frac{|\boldsymbol{K}_i|^{-1/2}}{(2\pi)^{p_i/2}|\boldsymbol{D}|^{1/2}} \exp(-\frac{\zeta_i}{2})d\boldsymbol{s}, \\
&= \int_{s=-\infty}^{s=\boldsymbol{\gamma_c}-\boldsymbol{X}_i\boldsymbol{\beta}} \frac{|\boldsymbol{K}_i|^{-1/2}}{(2\pi)^{p_i/2}|\boldsymbol{D}|^{1/2}} \exp\Big\{ -\frac{1}{2}\big[s'(I - \boldsymbol{Z}_i\boldsymbol{K}_i^{-1}\boldsymbol{Z}_i')s\big]\Big\}d\boldsymbol{s}, \\
&= \int_{s=-\infty}^{s=\boldsymbol{\gamma_c}-\boldsymbol{X}_i\boldsymbol{\beta}} \frac{|\boldsymbol{K}_i|^{-1/2}}{(2\pi)^{p_i/2}|\boldsymbol{D}|^{1/2}} \exp\Big\{ -\frac{s'Ls}{2}\Big\}d\boldsymbol{s}, \\
&= \frac{|\boldsymbol{K}_i|^{-1/2}}{|\boldsymbol{L}_i|^{1/2}|\boldsymbol{D}|^{1/2}} \Phi(\gamma_c - \boldsymbol{X}_i\boldsymbol{\beta}; \boldsymbol{L}_i^{-1}),
\end{aligned}$$

where we have written $\boldsymbol{L}_i = I - \boldsymbol{Z}_i\boldsymbol{K}_i^{-1}\boldsymbol{Z}_i'$.

Now, consider:

$$\begin{bmatrix} \boldsymbol{K}_i & \boldsymbol{Z}_i' \\ \boldsymbol{Z}_i & I \end{bmatrix}.$$

Then

$$\begin{aligned}
|\boldsymbol{K}_i| \cdot |I - \boldsymbol{Z}_i\boldsymbol{K}_i^{-1}\boldsymbol{Z}_i'| &= |I| \cdot |\boldsymbol{K}_i - \boldsymbol{Z}_i'I\boldsymbol{Z}_i| \\
|\boldsymbol{K}_i||\boldsymbol{L}_i| &= |\boldsymbol{D}^{-1} + \boldsymbol{Z}_i'\boldsymbol{Z}_i - \boldsymbol{Z}_i'\boldsymbol{Z}_i| \\
|\boldsymbol{K}_i||\boldsymbol{L}_i||\boldsymbol{D}| &= 1.
\end{aligned}$$

This result produces

$$f(\boldsymbol{y}_i \leqslant \boldsymbol{c}) = \Phi(\boldsymbol{\gamma}_c - \boldsymbol{X}_i\boldsymbol{\beta}; \boldsymbol{L}_i^{-1})$$

B. Computation of the marginal joint longitudinal normal-ordinal(probit) model

Let us derive (3.5) by integrating out the the random effects $\boldsymbol{\xi}_i$ of the joint density of $\boldsymbol{y}_{1i}$, $\boldsymbol{y}_{2i}$ and $\boldsymbol{\xi}_i$. Since the responses are independent conditional on the random effects $\boldsymbol{\xi}_i$, we can write:

$$
\begin{aligned}
&f(\boldsymbol{y}_{1i}, \boldsymbol{y}_{2i} \leqslant c) \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \frac{1}{(2\pi)^{(q+n_i)/2}|\boldsymbol{D}|^{1/2}|\boldsymbol{\Sigma}_{1i}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{\xi}_i'\boldsymbol{D}^{-1}\boldsymbol{\xi}_i\big]\Big\} \\
&\quad \exp\Big\{-\frac{1}{2}\big[(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)'\boldsymbol{\Sigma}_{1i}^{-1}(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)\big]\Big\} \prod_{k=1}^{p_i} \Phi(\gamma_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta} - \boldsymbol{z}_{2ik}'\boldsymbol{\xi}_i) d\boldsymbol{\xi}_i \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \frac{1}{(2\pi)^{(q+n_i)/2}|\boldsymbol{D}|^{1/2}|\boldsymbol{\Sigma}_{1i}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{\xi}_i'\boldsymbol{D}^{-1}\boldsymbol{\xi}_i\big]\Big\} \\
&\quad \exp\Big\{-\frac{1}{2}\big[(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)'\boldsymbol{\Sigma}_{1i}^{-1}(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)\big]\Big\} \Phi(\boldsymbol{\gamma_c} - \boldsymbol{X}_{2i}\boldsymbol{\beta} - \boldsymbol{Z}_{2i}\boldsymbol{\xi}_i) d\boldsymbol{\xi}_i \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{t=-\infty}^{t=\boldsymbol{\gamma_c}-\boldsymbol{X}_{2i}\boldsymbol{\beta}-\boldsymbol{Z}_{2i}\boldsymbol{\xi}_i} \frac{1}{(2\pi)^{(q+n_i+p_i)/2}|\boldsymbol{D}|^{1/2}|\boldsymbol{\Sigma}_{1i}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{\xi}_i'\boldsymbol{D}^{-1}\boldsymbol{\xi}_i\big]\Big\} \\
&\quad \exp\Big\{-\frac{1}{2}\big[(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)'\boldsymbol{\Sigma}_{1i}^{-1}(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)\big]\Big\} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{t}'\boldsymbol{t}\big]\Big\} d\boldsymbol{\xi}_i d\boldsymbol{t} \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{s=-\infty}^{s=\boldsymbol{\gamma_c}-\boldsymbol{X}_{ij}\boldsymbol{\beta}} \frac{1}{(2\pi)^{(q+n_i+p_i)/2}|\boldsymbol{D}|^{1/2}|\boldsymbol{\Sigma}_{1i}|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{\xi}_i'\boldsymbol{D}^{-1}\boldsymbol{\xi}_i\big]\Big\} \\
&\quad \exp\Big\{-\frac{1}{2}\big[(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)'\boldsymbol{\Sigma}_{1i}^{-1}(\boldsymbol{y}_{1i}-\boldsymbol{X}_{1i}\boldsymbol{\beta}-\boldsymbol{Z}_{1i}\boldsymbol{\xi}_i) + (\boldsymbol{s}-\boldsymbol{Z}_{2i}\boldsymbol{\xi}_i)'(\boldsymbol{s}-\boldsymbol{Z}_{2i}\boldsymbol{\xi}_i)\big]\Big\} d\boldsymbol{\xi}_i d\boldsymbol{s} \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma_c}-\boldsymbol{X}_{2i}\boldsymbol{\beta}} \frac{1}{(2\pi)^{(q+n_i+p_i)/2}|\boldsymbol{D}|^{1/2}|\boldsymbol{\Sigma}_i|^{1/2}} \exp\Big\{-\frac{1}{2}\big[\boldsymbol{\xi}_i-\boldsymbol{u}_i)'\boldsymbol{K}_i^{-1}(\boldsymbol{\xi}_i-\boldsymbol{u}_i)+\nu_i\big]\Big\} d\boldsymbol{\xi}_i \mathrm{d}\boldsymbol{s}.
\end{aligned}
$$

Here, $\boldsymbol{\gamma_c}$ denotes the vector of thresholds for category c, which can, but is not necessary equal for each timepoint. In addition, the variable $\boldsymbol{t}$ is rewritten as $\boldsymbol{t} = \boldsymbol{s} - \boldsymbol{Z}_{2i}\boldsymbol{\xi}_i$ and $\boldsymbol{\eta}_i = \boldsymbol{y}_{1i} - \boldsymbol{X}_{1i}\boldsymbol{\beta}$. Next,

$$
\boldsymbol{\xi}_i'\boldsymbol{D}^{-1}\boldsymbol{\xi}_i + (\boldsymbol{\eta}_i - \boldsymbol{Z}_{1i}\boldsymbol{\xi}_i)'\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\eta}_i - \boldsymbol{Z}_{1i}\boldsymbol{\xi}_i) + (\boldsymbol{s}-\boldsymbol{Z}_{2i}\boldsymbol{\xi}_i)'(\boldsymbol{s}-\boldsymbol{Z}_{2i}\boldsymbol{\xi}_i) = (\boldsymbol{\xi}_i-\boldsymbol{u}_i)'\boldsymbol{K}_i^{-1}(\boldsymbol{\xi}_i-\boldsymbol{u}_i)+\nu_i,
$$

where

$$\boldsymbol{u}_i = -\boldsymbol{K}_i\boldsymbol{l}_i,$$
$$\boldsymbol{K}_i^{-1} = \boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i} + \boldsymbol{Z}'_{2i}\boldsymbol{Z}_{2i},$$
$$\boldsymbol{l}'_i = -\boldsymbol{\eta}'_i\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i} - \boldsymbol{s}'\boldsymbol{Z}_{2i},$$
$$\nu_i = \boldsymbol{\eta}'_i\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\eta}_i - \boldsymbol{l}'\boldsymbol{K}_i\boldsymbol{l} + \boldsymbol{s}'\boldsymbol{s}.$$

Further, integrating over the random effects results in

$$\begin{aligned}
&f(\boldsymbol{y}_{1i}, \boldsymbol{y}_{2i} \leqslant c)\\
&= \int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma_c}-\boldsymbol{X}_{2i}\boldsymbol{\beta}} \frac{|\boldsymbol{K}_i|^{1/2}}{|\boldsymbol{D}|^{1/2}|\boldsymbol{\Sigma}_i|^{1/2}(2\pi)^{(p_i+n_i)/2}}\exp\{-\frac{1}{2}\nu_i\}\mathrm{d}\boldsymbol{s}\\
&= \int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma_c}-\boldsymbol{X}_{2i}\boldsymbol{\beta}} \frac{|\boldsymbol{K}_i|^{1/2}|\boldsymbol{\xi}_i|^{1/2}}{|\boldsymbol{D}|^{1/2}(2\pi)^{(p_i+n_i)/2}|\boldsymbol{\Sigma}_i|^{1/2}|\boldsymbol{\xi}_i|^{1/2}}\exp\left\{-\frac{1}{2}\Big[(\boldsymbol{s}-\boldsymbol{\alpha}_i)'\boldsymbol{\xi}_i^{-1}(\boldsymbol{s}-\boldsymbol{\alpha}_i)+c_i\Big]\right\}\mathrm{d}\boldsymbol{s},
\end{aligned}$$

where $\boldsymbol{\nu}_i = (s-\boldsymbol{\alpha}_i)'\boldsymbol{\xi}_i^{-1}(s-\boldsymbol{\alpha}_i) + c_i$, with

$$\boldsymbol{\alpha}_i = \boldsymbol{\xi}_i\boldsymbol{Z}_{2i}\boldsymbol{K}_i\boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\eta}_i,$$
$$\boldsymbol{\xi}_i^{-1} = \boldsymbol{I} - \boldsymbol{Z}_{2i}\boldsymbol{K}_i\boldsymbol{Z}'_{2i},$$
$$c_i = -\boldsymbol{\alpha}'_i\boldsymbol{\xi}_i^{-1}\boldsymbol{\alpha}_i + \boldsymbol{\eta}'_i(\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}\boldsymbol{K}_i\boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1})\boldsymbol{\eta}_i.$$

Next, denote $\boldsymbol{u} = \boldsymbol{s} - \boldsymbol{\alpha}_i$, which produces

$$f(\boldsymbol{y}_{1i}, \boldsymbol{y}_{2i} \leqslant c) = \frac{|\boldsymbol{K}_i|^{1/2}|\boldsymbol{\xi}_i|^{1/2}}{|\boldsymbol{D}|^{1/2}(2\pi)^{n_i/2}|\boldsymbol{\Sigma}_i|^{1/2}}\exp(\frac{-c_i}{2})\Phi(\boldsymbol{\gamma_c} - \boldsymbol{X}_{2i}\boldsymbol{\beta} - \boldsymbol{\alpha}_i, \boldsymbol{\xi}_i).$$

Now, consider:

$$\begin{bmatrix}\boldsymbol{K}_i^{-1} & \boldsymbol{Z}'_{2i}\\ \boldsymbol{Z}_{2i} & \boldsymbol{I}\end{bmatrix}$$

Then

$$|\boldsymbol{K}_i|^{-1}\cdot|\boldsymbol{I} - \boldsymbol{Z}_{2i}\boldsymbol{K}_i\boldsymbol{Z}'_{2i}| = |\boldsymbol{I}|\cdot|\boldsymbol{K}_i^{-1} - \boldsymbol{Z}'_{2i}\boldsymbol{Z}_{2i}|$$
$$|\boldsymbol{K}_i|^{-1}|\boldsymbol{\xi}_i|^{-1} = |\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}|$$
$$|\boldsymbol{K}_i|^{1/2}|\boldsymbol{\xi}_i|^{1/2} = |\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}|^{-1/2}.$$

Applying the latter results in the following equation:

$$f(\boldsymbol{y}_{1i}, \boldsymbol{y}_{2i} \leqslant c) = \frac{|\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}|^{-1/2}}{|\boldsymbol{D}|^{1/2}(2\pi)^{n_i/2}|\boldsymbol{\Sigma}_i|^{1/2}} \exp(\frac{-c_i}{2})\Phi_{p_i}(\boldsymbol{\gamma_c} - \boldsymbol{X}_{2i}\boldsymbol{\beta} - \boldsymbol{\alpha}_i, \boldsymbol{\xi}_i).$$

Next, write

$$\begin{aligned} c_i &= \boldsymbol{\eta}'_i(\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}\boldsymbol{K}_i(\boldsymbol{K}_i^{-1} + \boldsymbol{Z}'_{2i}\boldsymbol{\xi}_i\boldsymbol{Z}_{2i})\boldsymbol{K}_i\boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1})\boldsymbol{\eta}_i, \\ c_i &= \boldsymbol{\eta}'_i\boldsymbol{W}_i^{-1}\boldsymbol{\eta}_i, \\ \boldsymbol{W}_i^{-1} &= \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}\boldsymbol{K}_i[\boldsymbol{K}_i^{-1} + \boldsymbol{Z}'_{2i}\boldsymbol{\xi}_i\boldsymbol{Z}_{2i}]\boldsymbol{K}_i\boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}. \end{aligned}$$

Further, consider

$$\begin{bmatrix} -\boldsymbol{D}^{-1} & \boldsymbol{Z}'_{1i} \\ \boldsymbol{Z}_{1i} & \boldsymbol{I} \end{bmatrix}.$$

Hence,

$$|\boldsymbol{D}^{-1}| \cdot |\boldsymbol{\Sigma}_i + \boldsymbol{Z}_{1i}\boldsymbol{D}\boldsymbol{Z}'_{1i}| = |\boldsymbol{\Sigma}_i| \cdot |\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}|.$$

Hence,

$$\frac{1}{|\boldsymbol{D}|^{1/2}|\boldsymbol{\Sigma}_i|^{1/2}|\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}|^{1/2}} = \frac{1}{|\boldsymbol{\Sigma}_i + \boldsymbol{Z}_{1i}\boldsymbol{D}\boldsymbol{Z}'_{1i}|^{1/2}}.$$

Applying this result results in

$$f(\boldsymbol{y}_{1i}, \boldsymbol{y}_{2i} \leqslant c) = \frac{|\boldsymbol{W}_i|^{1/2}}{|\boldsymbol{V}_i|^{1/2}}\phi(\boldsymbol{X}_{1i}\boldsymbol{\beta}; \boldsymbol{W}_i)\Phi(\boldsymbol{\gamma_c} - \boldsymbol{X}_{2i}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \boldsymbol{\xi}_i).$$

Now, consider

$$\begin{aligned} \boldsymbol{D}^{-1} + \boldsymbol{Z}'_{2i}\boldsymbol{Z}_{2i} + &\boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i} - \boldsymbol{Z}'_{2i}\boldsymbol{Z}_{2i} = \boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i} \\ \boldsymbol{D}^{-1} + \boldsymbol{Z}'_{2i}\boldsymbol{Z}_{2i} + &\boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i} - \boldsymbol{Z}'_{2i}(\boldsymbol{I} - \boldsymbol{Z}_{2i}\boldsymbol{K}_i\boldsymbol{Z}'_{2i} + \boldsymbol{Z}_{2i}\boldsymbol{K}_i\boldsymbol{Z}'_{2i})^{-1}\boldsymbol{Z}_{2i} = \boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}. \end{aligned}$$

Inserting $\boldsymbol{\xi}_i^{-1} = \boldsymbol{I} - \boldsymbol{Z}_{2i}\boldsymbol{K}_i\boldsymbol{Z}'_{2i}$ and $\boldsymbol{K}_i^{-1} = \boldsymbol{D}^{-1} + \boldsymbol{Z}'_{2i}\boldsymbol{Z}_{2i} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}$ results in

$$\boldsymbol{K}_i^{-1} - \boldsymbol{Z}'_{2i}(\boldsymbol{\xi}_i^{-1} + \boldsymbol{Z}_{2i}\boldsymbol{K}_i\boldsymbol{Z}'_{2i})^{-1}\boldsymbol{Z}_{2i} = \boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{1i}.$$

Next, taking the inverse of both sides results in

$$\boldsymbol{K}_i + \boldsymbol{K}_i \boldsymbol{Z}'_{2i} \boldsymbol{\xi}_i \boldsymbol{Z}_{2i} \boldsymbol{K}_i = (\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{1i})^{-1}$$

$$\boldsymbol{K}_i [\boldsymbol{K}_i^{-1} + \boldsymbol{Z}'_{2i} \boldsymbol{\xi}_i \boldsymbol{Z}_{2i}] \boldsymbol{K}_i = (\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{1i})^{-1}$$

$$\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{1i} \boldsymbol{K}_i [\boldsymbol{K}_i^{-1} + \boldsymbol{Z}'_{2i} \boldsymbol{\xi}_i \boldsymbol{Z}_{2i}] \boldsymbol{K}_i \boldsymbol{Z}'_{1i} \boldsymbol{\Sigma}_i^{-1} = \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{1i} (\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{1i})^{-1} \boldsymbol{Z}'_{1i} \boldsymbol{\Sigma}_i^{-1}.$$

Since $V^{-1} = \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{1i} (\boldsymbol{D}^{-1} + \boldsymbol{Z}'_{1i} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{1i})^{-1} \boldsymbol{Z}'_{1i} \boldsymbol{\Sigma}_i^{-1}$, this results in

$$\boldsymbol{V}_i^{-1} = \boldsymbol{W}_i^{-1}.$$

Applying this result produces

$$f(\boldsymbol{y}_{1i}, \boldsymbol{y}_{2i} \leqslant c) = \phi(\boldsymbol{X}_{1i} \boldsymbol{\beta}; \boldsymbol{V}_i) \Phi(\boldsymbol{\gamma}_{\boldsymbol{c}} - \boldsymbol{X}_{2i} \boldsymbol{\beta} - \boldsymbol{\alpha}_i; \boldsymbol{\xi}_i).$$

C. Conditional distribution of a subvector of the continuous response given the ordinal response(s) and continuous response(s)

C.1 *Expected value*

Let us derive (3.7), the conditional expected value of the continuous subvector $\widetilde{\boldsymbol{Y}}_{ci}^{a}$=$(Y_{ci}^{a_1}Y_{ci}^{a_2},...,Y_{ci}^{a_{n_a}})$ given a distinct subvector of continuous responses $\widetilde{\boldsymbol{Y}}_{ci}^{b}$=$(Y_{ci}^{b_1}Y_{ci}^{b_2},...,Y_{ci}^{b_{n_b}})$ and the ordinal subvector $\widetilde{\boldsymbol{Y}}_{\boldsymbol{bi}}$ of length $\widetilde{p}_i$. Subvectors and submatrices will be indicated with superscripts; the superscript a denotes the rows a_1 until a_{n_a} and the superscript b denotes the rows b_1 until b_{n_b}. Analogously, the superscript bb denotes the submatrix with rows b_1 until b_{n_b} and columns b_1 until b_{n_b}. The superscript ab denotes the submatrix with rows a_1 until a_{n_a} and columns b_1 until b_{n_b}. Lastly, the superscript $.b$ denotes the columns b_1 until b_{n_b} from a matrix.

The conditional expected value of $\widetilde{\boldsymbol{Y}}_{ci}^{a}$ can be calculated via integrating $\widetilde{\boldsymbol{Y}}_{ci}^{a}$ out of the product of $\widetilde{\boldsymbol{Y}}_{ci}^{a}$ with the conditional density

$$
\begin{aligned}
&E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}|\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} = \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}] \\
&= \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=\infty} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \frac{\phi(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}; \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}; \boldsymbol{V}_i)\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \boldsymbol{B}_i)}{f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}})} \Bigg\} d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \qquad \text{(C.1)} \\
&= \frac{1}{c} \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i \begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\widetilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\
&\quad \exp\Bigg\{ -\frac{1}{2}\Bigg((\begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})' \boldsymbol{V}_i^{-1} (\begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) +' \boldsymbol{B}_i^{-1} \Bigg) \Bigg\} d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} d \\
&= \frac{1}{c} \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\widetilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\
&\quad \exp\Bigg\{ -\frac{1}{2}\Bigg((\begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})' \boldsymbol{V}_i^{-1} (\begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + (-\boldsymbol{H}_i \begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix})' \boldsymbol{B}_i^{-1} (-\boldsymbol{H}_i \begin{bmatrix} \widetilde{y}_{ci}^{a} \\ \widetilde{y}_{ci}^{b} \end{bmatrix}) \Bigg) \Bigg\} d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} d \\
&= \frac{1}{c} \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}}=\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\widetilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\
&\quad \exp\Bigg\{ -\frac{1}{2}\Bigg((\begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix} - \boldsymbol{u}_i)' \boldsymbol{E}_i^{-1} (\begin{bmatrix} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} \\ \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} \end{bmatrix} - \boldsymbol{u}_i) + O_i \Bigg) \Bigg\} d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} d,
\end{aligned}
$$

where

$$c = f(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}), \text{the marginal joint distribution}$$
$$= -\boldsymbol{H}_i \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}.$$

Further,

$$\begin{aligned}
&(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + (-\boldsymbol{H}_i\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix})'\boldsymbol{B}_i^{-1}(-\boldsymbol{H}_i\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix}) = \\
&\qquad \left(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \boldsymbol{u}_i\right)'\boldsymbol{E}_i^{-1}(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \boldsymbol{u}_i) + O_i,
\end{aligned}$$

with

$$\begin{aligned}
\boldsymbol{E}_i^{-1} &= \boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{V}_i^{-1}\\
l_i' &= -\boldsymbol{s}'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}'\boldsymbol{V}_i^{-1}\\
O_i &= \boldsymbol{s}'\boldsymbol{B}_i^{-1}\boldsymbol{s} + \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - (-\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{s} - \boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(-\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}s - \boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\\
\boldsymbol{u}_i &= -\boldsymbol{E}_i\boldsymbol{l}_i.
\end{aligned}$$

Integrating over $\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}$ results in

$$\begin{aligned}
&E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\\
&= \frac{1}{c}\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\widetilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\\
&\quad \left((\boldsymbol{u}_i^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b))\phi(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \boldsymbol{u}_i^b, \boldsymbol{E}_i^{bb})\right)\exp\left\{-\frac{1}{2}O_i\right\}d\\
&= \frac{1}{c}\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(\widetilde{p}_i + n_b)}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i||\boldsymbol{E}_i^{bb}|}}(\boldsymbol{u}_i^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b))\\
&\quad \exp\left\{-\frac{1}{2}\left((-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(-\boldsymbol{F}_i) + G_i\right)\right\}d.
\end{aligned} \tag{C.2}$$

where we have substituted $O_i + (\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b)'(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b) = (\boldsymbol{s} - \boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(\boldsymbol{s} - \boldsymbol{F}_i) + G_i$ with

$$\begin{aligned}
\boldsymbol{T}_i^{-1} &= (\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b + \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})\\
\boldsymbol{F}_i &= \boldsymbol{T}_i\Big((\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}_1)^b) + (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\Big)\\
G_i &= \Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}_1)^b\Big)'(\boldsymbol{E}_i^{bb})^{-1}\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}_1)^b\Big) - \boldsymbol{F}_i'\boldsymbol{T}_i^{-1}\boldsymbol{F}_i +\\
&\quad (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) - (\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}).
\end{aligned}$$

Integrating over results in

$$\begin{aligned}
E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}] &= \frac{e^{-0.5G_i}}{c(2\pi)^{\frac{n_b}{2}}}\frac{\sqrt{|\boldsymbol{E}_i||\boldsymbol{T}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i||\boldsymbol{E}_i^{bb}|}}\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i) \quad \text{(C.3)}\\
&\quad \Bigg\{\Big((\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\Big)\\
&\quad + \Big((\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i\boldsymbol{B}_i^{-1})^b\Big)\\
&\quad \times \Big(\boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big] + \boldsymbol{F}_i\Big)\Bigg\},
\end{aligned}$$

where the last factor equals the expected value of the truncated normal distribution with variance $\boldsymbol{T}_i$, mean $\boldsymbol{F}_i$ and limits $]-\infty; \boldsymbol{o} + \boldsymbol{F}_i]$. More specifically,

$$\begin{aligned}
\boldsymbol{o} &= \boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i\\
F_i(x_i) &= \int_{-\infty}^{o_1}\dots\int_{-\infty}^{o_{i-1}}\int_{-\infty}^{o_{i+1}}\dots\int_{-\infty}^{o_{\widetilde{p}_i}}\varphi(x_1, ...x_{i-1}, x, x_{i+1}, ...x_{\widetilde{p}_i})dx_{\widetilde{n}_i}, ..dx_{i+1}dx_{i-1}...dx_1,\\
\varphi(x) &= \begin{cases} \frac{\phi(x, \boldsymbol{T}_i)}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i, \boldsymbol{T}_i)}, & \text{for } \boldsymbol{x} \leqslant \boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Now consider,

$$\boldsymbol{T}_i^{-1} = (\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b} + \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})$$

$$\boldsymbol{B}_i^{-1}\boldsymbol{H}_i = \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}$$

Further, define

$$\boldsymbol{M}_a = \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}$$

$$\boldsymbol{M}_b = \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}$$

Since $\boldsymbol{\Sigma}_i = \sigma_i^2\boldsymbol{I}$ and the fact that $\widetilde{\boldsymbol{Z}}_{ci}$ and $\widetilde{\boldsymbol{Z}}_{bi}$ are design matrices

$$(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'} = \boldsymbol{M}_b\boldsymbol{E}^{bb} + \boldsymbol{M}_a\boldsymbol{E}^{ab}$$

As a consequence,

$$\begin{aligned}\boldsymbol{T}_i^{-1} = {} & \boldsymbol{M}_b\boldsymbol{E}^{bb}\boldsymbol{M}_b' + \boldsymbol{M}_b\boldsymbol{E}^{ab}\boldsymbol{M}_a' + \boldsymbol{M}_a\boldsymbol{E}^{ab}\boldsymbol{M}_b' + \\ & \boldsymbol{M}_a\boldsymbol{E}^{ab}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}^{ba}\boldsymbol{M}_a + \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})\end{aligned}$$

Further, by using the inverse of partitioned matrices,

$$(\boldsymbol{E}_i^{bb})^{-1} = (\boldsymbol{E}_i^{-1})^{bb} - (\boldsymbol{E}_i^{-1})^{ba}((\boldsymbol{E}_i^{-1})^{aa})^{-1}(\boldsymbol{E}_i^{-1})^{ab},$$

we get the following resullt

$$\begin{aligned}\boldsymbol{T}_i^{-1} = {} & \boldsymbol{M}_b\boldsymbol{E}^{bb}\boldsymbol{M}_b' + \boldsymbol{M}_b\boldsymbol{E}^{ab}\boldsymbol{M}_a' + \boldsymbol{M}_a\boldsymbol{E}^{ab}\boldsymbol{M}_b' + \boldsymbol{M}_a\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{-1})^{bb}\boldsymbol{E}_i^{ba}\boldsymbol{M}_a' - \\ & \boldsymbol{M}_a\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{-1})^{ba}((\boldsymbol{E}_i^{-1})^{aa})^{-1}(\boldsymbol{E}_i^{-1})^{ab}\boldsymbol{E}_i^{ba}\boldsymbol{M}_a' + \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}).\end{aligned}$$

Next, by the use that $\boldsymbol{B}_i^{-1} = \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'$ and $(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}) = [\boldsymbol{M}_b \;\; \boldsymbol{M}_a]$

$$\begin{aligned}\boldsymbol{T}_i^{-1} &= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}' + \boldsymbol{M}_a\left(\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}_i^{ba} - \boldsymbol{E}_i^{aa}\right)\boldsymbol{M}_a' \\ &= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}' - \boldsymbol{M}_a\left((\boldsymbol{E}_i^{-1})^{aa}\right)^{-1}\boldsymbol{M}_a',\end{aligned}$$

where we used the inverse of a partioned matrix. By again substituting $\boldsymbol{M}_a = \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}$

$$\begin{aligned}\boldsymbol{T}_i^{-1} &= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\Bigg[\boldsymbol{K}_i + \boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\Bigg((\boldsymbol{E}_i^{-1})^{aa}\Bigg)^{-1}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a}\boldsymbol{K}_i\Bigg]\widetilde{\boldsymbol{Z}}_{bi}' \\ &= \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\Bigg[\boldsymbol{W}_i\Bigg]\widetilde{\boldsymbol{Z}}_{bi}'.\end{aligned}$$

Now consider

$$\boldsymbol{W}_i^{-1} = \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\Bigg[(\boldsymbol{E}_i^{-1})^{aa} + (\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1})'\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\Bigg]^{-1}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a}$$

and

$$\begin{aligned}\boldsymbol{E}_i^{-1} &= \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Bigg(\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i - (\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\Bigg)\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1} \\ &= \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Bigg(\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i + \\ &\quad \boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{D}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i - (\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\Bigg)\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1} \\ &= \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Bigg(-\boldsymbol{K}_i\Bigg)\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1},\end{aligned}$$

where we substituted $\boldsymbol{B}_i = \boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'$ and $\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} = \boldsymbol{K}_i^{-1} - \boldsymbol{D}^{-1} - \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}$. As a consequence,

$$\begin{aligned}\boldsymbol{W}_i^{-1} &= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\Bigg[(\boldsymbol{\Sigma}_i^{aa})^{-1} - (\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1} + \\ &\quad (\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1})'\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\Bigg]^{-1}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a} \\ &= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\Bigg[(\boldsymbol{\Sigma}_i^{aa})^{-1}\Bigg]^{-1}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a} \\ &= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a} \\ &= \boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} - \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a} \\ &= \boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} + \widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\end{aligned}$$

As a result,

$$\boldsymbol{T}_i^{-1} = \boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\Bigg[\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} + \widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}\Bigg]^{-1}\widetilde{\boldsymbol{Z}}_{bi}', \qquad \text{(C.4)}$$

which equals $(\boldsymbol{B}_i^*)^{-1}$, the inverse of the $\boldsymbol{B}_i$ matrix of the joint density $f(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}})$.

Next, consider

$$\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i = \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{T}_i\Big((\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + (\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^b_{ci} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\Big) \tag{C.5}$$

Further,

$$\begin{aligned}
(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1} &= \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1} \\
&= \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1} \\
&\quad \Big\{ - \widetilde{\boldsymbol{Z}}^a_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}^{b'}_{ci} \\
&\quad \big[\boldsymbol{\Sigma}_i^{bb} - \widetilde{\boldsymbol{Z}}^b_{ci}(-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci})^{-1}\widetilde{\boldsymbol{Z}}^{b'}_{ci}\big]^{-1}\Big\} \\
&= \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}^a_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}^{b'}_{ci} \\
&\quad \big[(\boldsymbol{\Sigma}_i^{bb})^{-1} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\big] \\
&= \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}^a_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1},
\end{aligned}$$

where $\boldsymbol{K}_i^* = (\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} + \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci})^{-1}$ and we substituted $\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci} = (\boldsymbol{K}_i^*)^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}'_{bi} - \boldsymbol{D}^{-1}$ and $\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} = -\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}'_{bi} - \boldsymbol{D}^{-1}$.

Next, we rewrite

$$\boldsymbol{Z}^{a'}_{ci}(\boldsymbol{\Sigma}_i^{aa})^{-1}\widetilde{\boldsymbol{Z}}^a_{ci} = \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} - \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci}$$

As a result,

$$\begin{aligned}
(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1} &= \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} - \widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^b_{ci})\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} \\
&= \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1} - (\boldsymbol{K}_i^*)^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} + \boldsymbol{D}^{-1})\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1} \\
&= \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}
\end{aligned}$$

Hence,

$$-\boldsymbol{T}_i(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}\widetilde{\boldsymbol{y}}_{ci}^b = -\boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b, \tag{C.6}$$

where $\boldsymbol{H}_i^*$ equals the $\boldsymbol{H}_i$ matrix of the joint density $f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}})$. Next, consider again (C.5)

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b &= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{T}_i\Bigg((\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad (\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\Bigg) \\
&= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{B}_i^*(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\Big(\boldsymbol{\Sigma}_i - \widetilde{\boldsymbol{Z}}_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\Big)\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\Big(\widetilde{\boldsymbol{Z}}_{ci}\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\Big)\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\big[-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b,
\end{aligned}$$

where we substituted $\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} = \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}$. Further,

$$\begin{aligned}
(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b &= \boldsymbol{E}_i^{bb}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \boldsymbol{E}_i^{ba}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \boldsymbol{E}_i^{bb}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \boldsymbol{E}_i^{ba}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a \\
&= \Big(\boldsymbol{\Sigma}_i^{bb} - \widetilde{\boldsymbol{Z}}_{ci}^b\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}\Big)(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
&\quad \Big(\boldsymbol{\Sigma}_i^{ba} - \widetilde{\boldsymbol{Z}}_{ci}^b\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}\Big)(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
&\quad \Big(\boldsymbol{\Sigma}_i^{bb} - \widetilde{\boldsymbol{Z}}_{ci}^b\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}\Big)(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
&\quad \Big(\boldsymbol{\Sigma}_i^{ba} - \widetilde{\boldsymbol{Z}}_{ci}^b\big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}\Big)(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a.
\end{aligned} \tag{C.7}$$

As a result,

$$
\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b = {} & \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\big[-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1}\big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a \\
= {} & \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a - \\
& \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\Big[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Big]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a - \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a,
\end{aligned}
$$

where we substituted $\widetilde{\boldsymbol{Z}}_{ci}^{b'}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b} = (\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{b'}\widetilde{\boldsymbol{Z}}_{ci}^{b} - \boldsymbol{D}^{-1}$. Next, consider

$$
\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{ab}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a = \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^{a}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})
$$

and

$$
\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b = \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^{b}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})
$$

As a consequence,

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b &= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\left[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right]^{-1}\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\left[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{V}_i^{-1})^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\left[-\boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right]^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\widetilde{\boldsymbol{Z}}_{bi}((\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},
\end{aligned}$$

where we have rewritten $\boldsymbol{B}_i = I + \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'$ and $\boldsymbol{B}_i^* = I + \widetilde{\boldsymbol{Z}}_{bi}((\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi})^{-1}\widetilde{\boldsymbol{Z}}_{bi}'$. Next, by the substitution of

$$\begin{aligned}
\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1} \\
(\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^b + \boldsymbol{D}^{-1} \\
\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= \boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} - \boldsymbol{D}^{-1} \\
\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi} &= (\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^b - \boldsymbol{D}^{-1},
\end{aligned}$$

we have the following result

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b &= \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}_{ci}^{a'}\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^a\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}_{ci}^{a'}\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{ab}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*)\widetilde{\boldsymbol{Z}}_{ci}^{a'}\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{aa}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}.
\end{aligned}$$

Next, by substituting

$$\begin{aligned}
\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{ab} &= -(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}(\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci})\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb} \\
\left((\boldsymbol{\Sigma}_i + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{ci})^{-1}\right)^{aa} &= (\boldsymbol{\Sigma}_i^{-1})^{aa} - (\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}(\boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}'_{ci})\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa} \\
\boldsymbol{K}_i^* + \boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^* &= (\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1})^{-1} \\
\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} &= \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1} - \boldsymbol{D}^{-1}
\end{aligned}$$

As a result,

$$\begin{aligned}
\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b &= \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1})^{-1} \\
&\quad \widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
&\quad \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci} + \boldsymbol{D}^{-1})^{-1} \\
&\quad \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a \\
&= \widetilde{\boldsymbol{Z}}_{bi}(\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1})^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b,
\end{aligned}$$

where we substituted $\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a = \widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}$.

Next,

$$\begin{aligned}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i + \boldsymbol{H}_i^*\widetilde{\boldsymbol{y}}_{ci}^b &= \Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^* - \widetilde{\boldsymbol{Z}}_{bi}\big((\boldsymbol{K}_i^*)^{-1}\big)^{-1} + \\ &\quad \widetilde{\boldsymbol{Z}}_{bi}\big[(\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\big]^{-1}\Big)\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\ &= \Big(\Big[\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\big[(\boldsymbol{K}_i^*)^{-1} - \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\big]^{-1}\widetilde{\boldsymbol{Z}}'_{bi}\Big]\widetilde{\boldsymbol{Z}}_{bi} \\ &\quad \Big[\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b} + \boldsymbol{D}^{-1} + \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\Big]^{-1}\Big)\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\ &= \boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\ &= \boldsymbol{H}_i^*(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b.\end{aligned} \tag{C.8}$$

Combining (C.4),(C.6) and (C.8), we find that

$$\Phi(\boldsymbol{\gamma_c} - (\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta})^b - \boldsymbol{H}_i^*(\widetilde{\boldsymbol{y}}_{ci}^b - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b); \boldsymbol{B}_i^*) = \Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i). \tag{C.9}$$

Now consider,

$$\begin{aligned}G_i &= \Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)'(\boldsymbol{E}_i^{bb})^{-1}\Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big) - \boldsymbol{F}'_i\boldsymbol{T}_i^{-1}\boldsymbol{F}_i + \\ &\quad (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) - (\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\ &= -\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\widetilde{\boldsymbol{y}}_{ci}^b - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b) + (\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\Big)'\boldsymbol{T}_i \\ &\quad \Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\widetilde{\boldsymbol{y}}_{ci}^b - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b) + (\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\Big) \\ &\quad + \Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)'(\boldsymbol{E}_i^{bb})^{-1}\Big(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big) \\ &\quad + (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) - (\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}).\end{aligned}$$

Next, we take the terms where $\widetilde{\boldsymbol{y}}_{ci}^{b}$ occurs twice from the latter equation

$$
\begin{aligned}
&-\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\Big)'\boldsymbol{T}_i\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\Big)+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\widetilde{\boldsymbol{y}}_{ci}^{b}\\
&=-\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\Big)'\boldsymbol{T}_i\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\Big)+\\
&\quad\widetilde{\boldsymbol{y}}_{ci}^{b'}\Big((\boldsymbol{E}_i^{-1})^{bb}-(\boldsymbol{E}_i^{-1})^{ba}((\boldsymbol{E}_i^{-1})^{aa})^{-1}(\boldsymbol{E}_i^{-1})^{ab}\Big)\widetilde{\boldsymbol{y}}_{ci}^{b}\\
&=\widetilde{\boldsymbol{y}}_{ci}^{b'}\Big\{-(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{B}_i^*\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}+(\boldsymbol{\Sigma}_i^{-1})^{bb}-(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}-\\
&\quad(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}+\\
&\quad(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\\
&\quad\Big(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\Big)^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\Big\}\widetilde{\boldsymbol{y}}_{ci}^{b}\\
&=\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big\{-\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}'\Big(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\big((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\big)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\Big)\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*-\boldsymbol{K}_i-\\
&\quad\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\boldsymbol{K}_i+\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\Big(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\Big)^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{a'}\\
&\quad(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\boldsymbol{K}_i\Big\}\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b},
\end{aligned}
$$

where we used the inverse of partitioned matrices $(\boldsymbol{E}_i^{bb})^{-1}=(\boldsymbol{E}_i^{-1})^{bb}-(\boldsymbol{E}_i^{-1})^{ba}((\boldsymbol{E}_i^{-1})^{aa})^{-1}(\boldsymbol{E}_i^{-1})^{ab}$ and substituted $\boldsymbol{B}_i^*=\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\big((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\big)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'$. After repeatedly using $\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}=\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{K}_i^{-1}+\boldsymbol{K}_i^{-1}$, we find that

$$
\begin{aligned}
&-\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\Big)'\boldsymbol{T}_i\Big(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\Big)+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\widetilde{\boldsymbol{y}}_{ci}^{b}\\
&=\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big\{\boldsymbol{K}_i^*-\Big((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\Big)^{-1}-\Big(\boldsymbol{K}_i^{-1}-\widetilde{\boldsymbol{Z}}_{ci}^{a'}(\boldsymbol{\Sigma}_i^{-1})^{aa}\widetilde{\boldsymbol{Z}}_{ci}^{a}\Big)^{-1}\Big\}\\
&\quad\widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\\
&=\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}+\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big\{\boldsymbol{K}_i^*-\Big((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\Big)^{-1}-\boldsymbol{K}_i^*\Big\}\widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\\
&=\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}-\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big(\boldsymbol{D}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\Big)^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{b}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^{b}\\
&=\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{V}_i^*)^{-1}\widetilde{\boldsymbol{y}}_{ci}^{b},
\end{aligned}
$$

where $(\boldsymbol{V}_i^*)^{-1}$ equals the inverse of the $\boldsymbol{V}_i$ matrix of the joint density $f(\widetilde{\boldsymbol{y}}_{ci}^b, \widetilde{\boldsymbol{y}}_{bi})$.

Next, consider the terms where $\widetilde{\boldsymbol{y}}_{ci}^b$ occurs once, at the start of the term,

$$
\begin{aligned}
&\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \\
&\qquad\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^b\right)'\boldsymbol{T}_i(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= \left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&- \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&+ \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a - \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&+ \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= \left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&- \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\right)^b\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&- \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\right)^b\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&- \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left((\boldsymbol{V}_i - \boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\boldsymbol{V}_i)\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\right)^b\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&+ \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{y}}_{ci}^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b - \widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b \\
&= -\widetilde{\boldsymbol{y}}_{ci}^{b'}(\boldsymbol{V}_i^*)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b,
\end{aligned}
$$

where we substituted

$$
\begin{aligned}
(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b = &\left(\boldsymbol{I} - \boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{ba}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
&\qquad\left(\boldsymbol{I} - \boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i)\right)^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b
\end{aligned}
$$

Lastly, consider the terms where $\widetilde{\boldsymbol{y}}_{ci}^{b}$ does not occur

$$
\begin{aligned}
&-\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b+\\
&\qquad\qquad\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i\\
&\qquad(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+\left((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)'\boldsymbol{T}_i\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)-\\
&\qquad\left((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)'\boldsymbol{T}_i\left((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\right)+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})-\\
&\qquad(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})+(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\\
=&-\left(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\right)'\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b+\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b'}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
=&(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{V}_i^*)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
=&(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{V}_i^*)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\\
&-(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\Big[-\boldsymbol{V}_i^{-1}+\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i+\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\Big]\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}+\\
&+(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\big)^{-1}\boldsymbol{H}_i\right)^b\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},
\end{aligned}
$$

where we used the results of the previous calculations.

Now consider,

$$\left(\boldsymbol{V}_i\boldsymbol{H}_i'(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i')^{-1}\boldsymbol{H}_i\right)^b=\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}')^{-1}\boldsymbol{H}_i$$

$$(\boldsymbol{E}_i^{bb})^{-1}=(\boldsymbol{\Sigma}_i^{-1})^{bb}-(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}$$

As a result,

$$\begin{aligned}
&\left(\boldsymbol{V}_i\boldsymbol{H}_i'(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i')^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}_i'(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i')^{-1}\boldsymbol{H}_i\right)^b\\
=&\left(\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}')^{-1}\boldsymbol{H}_i\right)'\left((\boldsymbol{\Sigma}_i^{-1})^{bb}-(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\right)\left(\widetilde{\boldsymbol{Z}}_{ci}^b\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}')^{-1}\boldsymbol{H}_i\right)\\
=&\boldsymbol{H}_i'\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\left((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{D}^{-1}\right)\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\boldsymbol{H}_i\\
&-\boldsymbol{H}_i'\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\left((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{D}^{-1}\right)\boldsymbol{K}_i^*\\
&\left((\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{D}^{-1}\right)\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\boldsymbol{H}_i\\
=&\boldsymbol{H}_i'\boldsymbol{H}_i-\boldsymbol{H}_i'\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\boldsymbol{H}_i-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i\\
=&\boldsymbol{H}_i'\boldsymbol{H}_i-\boldsymbol{H}_i'\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\boldsymbol{H}_i-\boldsymbol{H}_i'\left(\widetilde{\boldsymbol{Z}}_{bi}(\boldsymbol{K}_i^*)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'+\boldsymbol{I}-\boldsymbol{I}\right)\boldsymbol{H}_i\\
=&-\boldsymbol{H}_i'\left(\boldsymbol{I}+\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\boldsymbol{H}_i+\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\\
=&-\boldsymbol{H}_i'\boldsymbol{H}_i+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\left(\boldsymbol{D}^{-1}+\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}\right)^{-1}\boldsymbol{H}_i+\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\\
=&-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{B}_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}'\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}_i'\widetilde{\boldsymbol{Z}}_{bi}\left(\boldsymbol{D}^{-1}+\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}'\right)^{-1}\widetilde{\boldsymbol{Z}}_{bi}'\boldsymbol{H}_i+\boldsymbol{H}_i'(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i,
\end{aligned}$$

where we substituted $\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}\widetilde{\boldsymbol{Z}}_{ci}^{b'}=(\boldsymbol{K}_i^*)^{-1}-\widetilde{\boldsymbol{Z}}_{bi}'\widetilde{\boldsymbol{Z}}_{bi}-\boldsymbol{D}^{-1}$ and $\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'=\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}_{bi}'+\boldsymbol{I}-\boldsymbol{I}$.

Next,

$$
\begin{aligned}
&\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b'}(\boldsymbol{E}_i^{bb})^{-1}\left(\boldsymbol{V}_i\boldsymbol{H}'_i\big(\boldsymbol{B}_i+\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i\big)^{-1}\boldsymbol{H}_i\right)^{b}-\boldsymbol{H}'_i(\boldsymbol{B}^*_i)^{-1}\boldsymbol{H}_i\\
=&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i\\
&-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\\
&+\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\\
&-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}+\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i\\
&-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{bi}\boldsymbol{H}_i\\
=&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}-\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}\\
&+\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}+\boldsymbol{H}'_i\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}\\
&-\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\left(-\boldsymbol{K}_i^{-1}+\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\right)^{-1}\widetilde{\boldsymbol{Z}}'_{ci}\boldsymbol{V}_i^{-1}\\
=&-\boldsymbol{V}_i^{-1}+\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1},
\end{aligned}
$$

where we substituted

$$\begin{aligned}
\boldsymbol{V}_i^{-1} = & \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{'}\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}^{'}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{'}\boldsymbol{\Sigma}_i^{-1} \\
& -\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{bi}^{'}\widetilde{\boldsymbol{Z}}_{bi}\Bigg(\boldsymbol{K}_i^{-1} - \widetilde{\boldsymbol{Z}}_{bi}^{'}\widetilde{\boldsymbol{Z}}_{bi}\Bigg)^{-1}\widetilde{\boldsymbol{Z}}_{bi}^{'}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i\widetilde{\boldsymbol{Z}}_{ci}^{'}\boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}_i = & \boldsymbol{\Sigma}_i - \widetilde{\boldsymbol{Z}}_{ci}\Bigg(- \boldsymbol{K}_i^{-1} + \widetilde{\boldsymbol{Z}}_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{Z}}_{ci}\Bigg)^{-1}\widetilde{\boldsymbol{Z}}_{ci}^{'}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
& -\Bigg(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Bigg)^{'}\boldsymbol{T}_i(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'})(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b + \\
& \Bigg(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Bigg)^{'}\boldsymbol{T}_i(\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \Bigg((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\Bigg)^{'}\boldsymbol{T}_i\Bigg(\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}_{ci}^{b'}(\boldsymbol{\Sigma}_i^{-1})^{bb}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Bigg) - \\
& \Bigg((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\Bigg)^{'}\boldsymbol{T}_i\Bigg((\boldsymbol{B}_i^*)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\Bigg) + (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + \\
& (\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\
& = (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{b\prime}(\boldsymbol{V}_i^*)^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b.
\end{aligned}$$

Hence,

$$G_i = \Bigg(\widetilde{\boldsymbol{y}}_{ci}^{b'} - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Bigg)^{'}(\boldsymbol{V}_i^*)^{-1}\Bigg(\widetilde{\boldsymbol{y}}_{ci}^{b'} - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Bigg) \tag{C.10}$$

Now, consider

$$
\begin{aligned}
\frac{|\boldsymbol{E}_i||\boldsymbol{T}_i||\boldsymbol{V}_i^*|}{|\boldsymbol{V}_i||\boldsymbol{B}_i||\boldsymbol{E}_i^{bb}|} &= \frac{|(\boldsymbol{E}_i^{bb})^{-1}||\boldsymbol{V}_i^*|}{|\boldsymbol{V}_i||\boldsymbol{E}_i^{-1}||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|} \qquad (C.11)\\
&= \frac{|(\boldsymbol{\Sigma}_i^{bb})^{-1} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}||\boldsymbol{\Sigma}_i^{bb} + \widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|}{|\boldsymbol{V}_i||\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{V}_i^{-1}||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci} + (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}^*\widetilde{\boldsymbol{Z}}^{b}{}'_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|}{|\boldsymbol{V}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{I}_n||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} - (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}^{b'}_{ci} + \boldsymbol{\Sigma}_i^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci} + \boldsymbol{\Sigma}_i^{bb}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}^*\Big((\boldsymbol{K}^*)^{-1} - \widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi} - \boldsymbol{D}^{-1}\Big)\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|}{|\boldsymbol{I}_p + \boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'||\boldsymbol{B}_i||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} + (\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}|}{|\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi}||\boldsymbol{T}_i^{-1}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi}||\boldsymbol{I} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi}\widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi} - \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\Big((\boldsymbol{K}_i^*)^{-1} - \boldsymbol{D}^{-1} - \widetilde{\boldsymbol{Z}}^{b}{}'_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\Big)\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}\\
&= \frac{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|}{|\boldsymbol{I} + \widetilde{\boldsymbol{Z}}_{bi}\boldsymbol{D}\widetilde{\boldsymbol{Z}}^{b'}_{ci}(\boldsymbol{\Sigma}_i^{bb})^{-1}\widetilde{\boldsymbol{Z}}^{b}_{ci}\boldsymbol{K}_i^*\widetilde{\boldsymbol{Z}}'_{bi}|},\\
&= 1
\end{aligned}
$$

where we repeatedly used a familiar form of the Sylvester's identity: det(I+AB)=det(I+BA).

When we insert the results of (C.9), (C.10) and (C.11) in (C.3), the expected value simplifies to

$$
\begin{aligned}
E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}] = &\Big((\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\Big) \qquad (C.12)\\
&+ \Big((\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i\boldsymbol{B}_i^{-1})^b\Big)\\
&\times \Big(\boldsymbol{T}_i\big[-\boldsymbol{F}_1(o_1) \quad -\boldsymbol{F}_2(o_2) \quad \dots \quad -\boldsymbol{F}_p(o_p)\big] + \boldsymbol{F}_i\Big),
\end{aligned}
$$

C.2 *Prediction interval*

The prediction interval of a subvector of continuous responses conditional on both subvectors of the continuous response(s) and ordinal response(s) can be derived in analogy with Appendix D.2. The second central moment of the conditional distribution will be derived first, and next the standard errors of the transformed parameters will be computed.

The second central moment equals

$$
\begin{aligned}
&E\Big[\big(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)\big(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)'\Big] \\
&= \int_{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=\infty} \Big[\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - \Xi\Big]\Big[\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - \Xi\Big]' f(\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) d\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} \\
&= \int_{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=+\infty} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}{}' f(\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) - \Xi\Xi' f(\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) - \Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}{}'\Xi + \Xi\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}{}'\Big) \\
&\quad f(\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) d\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} \\
&= \int_{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=+\infty} \widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}{}' f(\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) d\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - \Xi\Xi' \\
&= \frac{1}{c}\int_{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i \begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}{}' \\
&\quad \exp\Bigg\{-\frac{1}{2}\Bigg(\big(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\big)'\boldsymbol{V}_i^{-1}\big(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\big) +' \boldsymbol{B}_i^{-1}\Bigg)\Bigg\} d\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} d - \Xi\Xi' \\
&= \frac{1}{c}\int_{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}{}' \\
&\quad \exp\Bigg\{-\frac{1}{2}\Bigg(\big(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\big)'\boldsymbol{V}_i^{-1}\big(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\big) + \big(-\boldsymbol{H}_i\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix}\big)'\boldsymbol{B}_i^{-1}\big(-\boldsymbol{H}_i\begin{bmatrix}\widetilde{y}^{a}_{ci}\\ \widetilde{y}^{b}_{ci}\end{bmatrix}\big)\Bigg) \\
&\quad \Bigg\} d\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} d - \Xi\Xi' \\
&= \frac{1}{c}\int_{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}=\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(n_a+n_b+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}{}' \\
&\quad \exp\Bigg\{-\frac{1}{2}\Bigg(\big(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \boldsymbol{u}_i\big)'\boldsymbol{E}_i^{-1}\big(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \boldsymbol{u}_i\big) + O_i\Bigg)\Bigg\} d\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} d - \Xi\Xi',
\end{aligned}
$$

where

$$\Xi = E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} | \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]$$

$$c = f(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}}), \text{the marginal joint distribution}$$

$$= -\boldsymbol{H}_i \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}.$$

Further,

$$\begin{aligned}(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + (-\boldsymbol{H}_i\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix})'\boldsymbol{B}_i^{-1}(-\boldsymbol{H}_i\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix})=\\ (\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \boldsymbol{u}_i)'\boldsymbol{E}_i^{-1}(\begin{bmatrix}\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}\\ \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}\end{bmatrix} - \boldsymbol{u}_i) + O_i,\end{aligned}$$

where $\boldsymbol{E}_i$,$\boldsymbol{l}_i$,O_i and $\boldsymbol{u}_i$ are defined in C.1.

Next, the integration over $\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}$ results in the following equation

$$\begin{aligned}&E\Big[\big(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - \Xi\big)\big(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}} - \Xi\big)'\Big]\\ &= \frac{1}{c}\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\widetilde{p}}{2}}} \frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\Big\{\boldsymbol{E}_i^{aa} - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}_i^{ba} +\\ &\Big(\boldsymbol{u}_i^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b)\Big)\\ &\times\Big(\boldsymbol{u}_i^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b)\Big)'\Big\}\phi(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \boldsymbol{u}_i^b, \boldsymbol{E}_i^{bb})\exp\{-\frac{1}{2}O_i\}d - \Xi\Xi'\\ &= \frac{1}{c}\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{n_b+\widetilde{p}}{2}}} \frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i||\boldsymbol{E}_i^{bb}|}}\Big\{\boldsymbol{E}_i^{aa} - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}_i^{ba} +\\ &\Big(\boldsymbol{u}_i^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b)\Big)\Big(\boldsymbol{u}_i^a + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - \boldsymbol{u}_i^b)\Big)'\Big\}\\ &\exp\Big\{-\frac{1}{2}\Big((-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(-\boldsymbol{F}_i) + G_i\Big)\Big\}d - \Xi\Xi'.\end{aligned}$$

Here we have rewritten $(\widetilde{y}^b_{ci} - \boldsymbol{u}_i^b)'(\boldsymbol{E}_i^{bb})^{-1}(\widetilde{y}^b_{ci} - \boldsymbol{u}_i^b) + O_i = (-\boldsymbol{F}_i)'T_i^{-1}(-\boldsymbol{F}_i) + G_i$, where $\boldsymbol{T}_i, G_i$ and $\boldsymbol{F}_i$ are defined in (C.2). Integrating over and implementing the results (C.9), (C.10) and

(C.11) results in

$$
\begin{aligned}
&E\Big[\big(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}-\Xi\big)\big(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}-\Xi\big)'\Big]\\
&=\boldsymbol{E}_i^{aa}-\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}_i^{ba}+(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a\Big(\boldsymbol{N}+\boldsymbol{J}\boldsymbol{J}'\Big)(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{a\prime} \qquad \text{(C.13)}\\
&+(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a\boldsymbol{J}((\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i)^{a'}+(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{a\prime}\\
&+(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^{a\prime}\\
&+\Bigg\{(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a\boldsymbol{J}\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)'-(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a\Big(\boldsymbol{N}+\boldsymbol{J}\boldsymbol{J}'\Big)(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b\prime}\\
&\quad+(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)'-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b\prime}\Bigg\}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}_i^{ba}\\
&\quad+\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\Bigg\{\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)((\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i\boldsymbol{E}_i)^{a'}-(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b\boldsymbol{J}((\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i\boldsymbol{E}_i)^{a'}\\
&\quad+\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{a\prime}-(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b\Big(\boldsymbol{N}+\boldsymbol{J}\boldsymbol{J}'\Big)(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{a\prime}\Bigg\}\\
&\quad+\boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\Bigg\{(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b\Big(\boldsymbol{N}+\boldsymbol{J}\boldsymbol{J}')\Big)(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b\prime}-\\
&\quad(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b\boldsymbol{J}\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)'\\
&\quad-\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)\boldsymbol{J}'(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^{b\prime}\\
&\quad+\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b\Big)\Big(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}-(\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\Big)'\Bigg\}(\boldsymbol{E}_i^{bb})^{-1}\boldsymbol{E}_i^{ba}-\Xi\Xi',
\end{aligned}
$$

with $\boldsymbol{J}$ as the expected value of the truncated multivariate normal density, and $\boldsymbol{N}$ is the second moment of the latter density. They can be implemented via the R package `tmvtnorm`. The following equations are derived from Manjunath and Wilhelm (2021). More specifically,

$$
\begin{aligned}
&\boldsymbol{d} = \boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},\\
&J = \boldsymbol{T}_i\big[-F_1(\boldsymbol{d}_1) \quad -F_2(\boldsymbol{d}_2) \quad \dots \quad -F_{\widetilde{p}_i}(\boldsymbol{d}_{\widetilde{p}_i})\big] + \boldsymbol{F}_i,\\
N_{i,j} &= T_{i\ i,j} + \sum_{k=1}^{\widetilde{p}_i} T_{i\ i,k}\frac{-T_{i\ j,k}\boldsymbol{d}_k F_k(\boldsymbol{d}_k)}{T_{i\ k,k}} + \sum_{k=1}^{\widetilde{p}_i} T_{i\ i,k}\sum_{q\neq k}\left(T_{i\ j,q} - \frac{T_{i\ k,q}T_{i\ j,k}}{T_{i\ k,k}}\right)\\
&\quad \cdot - F_{k,q}(\boldsymbol{d}_k, \boldsymbol{d}_q) - J_i J_k,\\
F_{k,q}(x,y) &= \int_{-\infty}^{\boldsymbol{d}_1} \dots \int_{-\infty}^{\boldsymbol{d}_{k-1}} \int_{-\infty}^{\boldsymbol{d}_{k+1}} .. \int_{-\infty}^{\boldsymbol{d}_{q-1}} \int_{-\infty}^{\boldsymbol{d}_{q+1}} \dots \int_{-\infty}^{\boldsymbol{d}_{\widetilde{n}_i}} \varphi(x,y,x_{-k-,-q})dx_{-k-,-q},\\
F_i(x_i) &= \int_{-\infty}^{\boldsymbol{d}_1} \cdots \int_{-\infty}^{\boldsymbol{d}_{i-1}} \int_{-\infty}^{\boldsymbol{d}_{i+1}} \cdots \int_{-\infty}^{\boldsymbol{d}_{\widetilde{p}_i}} \varphi(x_1, \dots x_{i-1}, x, x_{i+1}, \dots x_{\widetilde{p}_i})dx_{\widetilde{n}_i}, ..dx_{i+1}dx_{i-1}\dots dx_1,\\
\varphi(x) &= \begin{cases} \frac{\phi(x,\boldsymbol{F}_i,\boldsymbol{T}_i)}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i)}, & \text{for } \boldsymbol{x} \leqslant \boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta},\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Agresti (2002) utilized the delta method to determine the distribution of transformed maximum likelihood parameters. The distribution can be expressed as:

$$G(\hat{\boldsymbol{\theta}}) \to N\left(\boldsymbol{\theta}, \left(\frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)' Var(\hat{\boldsymbol{\theta}})\frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right),$$

Here, $\boldsymbol{\theta}$ denotes the parameter vector. To begin, we will outline the derivative of the expected value in relation to a coefficient β_{c2} associated with the predictor $\widetilde{\boldsymbol{X}}_{c2}$ for a continuous response. Subsequently, we will describe the derivation process for a coefficient β_{b2} linked to the predictor $\widetilde{\boldsymbol{X}}_{b2}$ for an ordinal response. We will then proceed to derive the gradients of the variance parameters. The derivative of the expected value (3.7) with respect to β_{c2}, an arbitrary coefficient of a predictor of the continuous response vector $\widetilde{\boldsymbol{X}}_{c2}$ is the following:

$$
\begin{aligned}
&\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} | \widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} = \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \beta_{c2}}\\
&= (\boldsymbol{E}_i\boldsymbol{V}_i^{-1})^a\widetilde{\boldsymbol{X}}_{c2i} - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_iV^{-1})^b\widetilde{\boldsymbol{X}}_{c2i}\\
&+ \Big((\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^a - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})^b\Big)\Big(\boldsymbol{\nu} + \boldsymbol{\delta}_i\Big),
\end{aligned}
$$

with

$$\boldsymbol{\delta}_i = \boldsymbol{T}_i \cdot \Big(-(\boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1})^{b\prime} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{c2i})^b + (\boldsymbol{H}_i' \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i (\boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{c2i}) \Big),$$

$$\boldsymbol{\Theta} = \sum_{k=1}^{\widetilde{p}} (\boldsymbol{H}_{ik} X_{12i} - \boldsymbol{\delta}_{ik}) \phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i X_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, \boldsymbol{T}_{kk})$$
$$\Phi\big[\phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}, \boldsymbol{T}_{-k|k}\big].$$

Further, $\boldsymbol{T}_i$ is partitioned as

$$\boldsymbol{T}_i = \begin{bmatrix} \boldsymbol{T}_{11}^{(k)} & \boldsymbol{T}_{c2}^{(k)} \\ \boldsymbol{T}_{2c}^{(k)} & \boldsymbol{T}_{kk} \end{bmatrix},$$

and $\boldsymbol{T}_{-k|k}$ is defined as

$$\boldsymbol{T}_{-k|k} = \boldsymbol{T}_{11} - \boldsymbol{T}_{c2} \boldsymbol{T}_{kk}^{-1} \boldsymbol{T}_{2c}. \tag{C.14}$$

In addition,

$$\boldsymbol{\nu} = \frac{\sum_{k=1}^{\widetilde{p}_i} (\boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{12i} - \boldsymbol{\delta}_i)_k g_k(o_k) - \boldsymbol{\Theta} \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p) \big]}{\Phi(\gamma - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i)},$$

$$g_k(\widetilde{\boldsymbol{X}}_k) = \int_{-\infty}^{o_1} \dots \int_{-\infty}^{o_{i-1}} \int_{-\infty}^{o_{i+1}} \dots \int_{-\infty}^{o_{\widetilde{p}_i}} [x_1..x_{k-1} o_k x_{k+1}..x_{\widetilde{p}_i}]' \varphi([x_1..x_{k-1} o_k x_{k+1}..x_{\widetilde{p}_i}]', \boldsymbol{T}_i) d\widetilde{\boldsymbol{X}}_{-k}$$

$$\varphi(x) = \begin{cases} \frac{\phi(x, \boldsymbol{T}_i)}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{T}_i)}, & \text{for } \boldsymbol{x} \leqslant \boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i, \\ 0, & \text{otherwise.} \end{cases}$$

The derivative of the expected value with respect to a coefficient β_{b2} of a predictor $\widetilde{\boldsymbol{X}}_{b2}$ of the ordinal response vector equals

$$\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{a}} | \widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}^{\boldsymbol{b}} = \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}^{\boldsymbol{b}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \beta_{b2}} = \Big((\boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1})^a - \boldsymbol{E}_i^{ab} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1})^b \Big)$$
$$\frac{\zeta - \Omega \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad \dots \quad -F_p(o_p) \big]}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i)},$$

with

$$\zeta = -\sum_{k=1}^{\widetilde{p}_i} \widetilde{X}_{b2ik}' g_k(\boldsymbol{o}_k),$$

$$\Omega = -\sum_{k=1}^{\widetilde{p}} X_{b2ik} \phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, \boldsymbol{T}_{kk}) \Phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}, \boldsymbol{T}_{-k|k}\big].$$

Next, the derivative of the expected value with respect to the threshold value γ_c of the ordinal response vector equals

$$\frac{\partial E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \gamma_c} = \Big((\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})^a - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})^b \Big) \frac{\eta - \omega\boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad \ldots \quad -F_p(o_p)\big]}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i)},$$

with

$$\eta = \sum_{k=1}^{\widetilde{p}_i} g_k(\boldsymbol{o}_k)$$
$$\omega = \sum_{k=1}^{\widetilde{p}} \phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, \boldsymbol{T}_{kk})\Phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}, \boldsymbol{T}_{-k|k}\big].$$

The derivative of the expected value with respect to an arbitrary component of $\boldsymbol{D}$, denoted by τ equals

$$\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \tau} &= (\boldsymbol{E}_i^*\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\boldsymbol{V}_i^*\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
&\quad \big[(\boldsymbol{E}_i^*)^{ab}(\boldsymbol{E}_i^{bb})^{-1} - \boldsymbol{E}_i^{ab}((\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i^*)^{bb}(\boldsymbol{E}_i^{bb})^{-1})\big] \\
&\quad \big[(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\big] + \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}\big[-\boldsymbol{E}_i^*\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\boldsymbol{V}_i^*\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\big]^b \\
&+ \Big((\boldsymbol{E}_i^*\boldsymbol{H}'_i\boldsymbol{B}_{\mathbf{i}}^{-1} + \boldsymbol{E}_i\boldsymbol{H}_i^{*'}\boldsymbol{B}_{\mathbf{i}}^{-1} - \boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1}\boldsymbol{B}_i^*\boldsymbol{B}_i^{-1})^a \\
&- \Big[(\boldsymbol{E}_i^*)^{ab}(\boldsymbol{E}_i^{bb})^{-1} - \boldsymbol{E}_i^{ab}((\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i^*)^{bb}(\boldsymbol{E}_i^{bb})^{-1})\Big](\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_{\mathbf{i}}^{-1})^b \\
&- \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i^*\boldsymbol{H}'_i\boldsymbol{B}_{\mathbf{i}}^{-1} + \boldsymbol{E}_i\boldsymbol{H}_i^{*'}\boldsymbol{B}_{\mathbf{i}}^{-1} - \boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_i^{-1}\boldsymbol{B}_i^*\boldsymbol{B}_i^{-1})^b \Big) \\
&\times \Big(\boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad \ldots \quad -F_p(o_p)\big] + \boldsymbol{F}_i \Big) \\
&+ \Big((\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_{\mathbf{i}}^{-1})^a - \boldsymbol{E}_i^{ab}(\boldsymbol{E}_i^{bb})^{-1}(\boldsymbol{E}_i\boldsymbol{H}'_i\boldsymbol{B}_{\mathbf{i}}^{-1})^b \Big)\boldsymbol{tr}^*.
\end{aligned}$$

To allow for a convenient solution for a general case, the following expression was evaluated numerically: $\boldsymbol{tr}^* = \frac{\partial \boldsymbol{T}_i \left[-F_1(o_1) \quad -F_2(o_2) \quad \ldots \quad -F_p(o_p) \right] + \boldsymbol{F}_i}{\partial \tau}$.In addition,

$$\boldsymbol{D}^* = \frac{\partial \boldsymbol{D}}{\partial \tau}$$

$$\boldsymbol{B}_i^* = \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^* \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}'_{bi} \boldsymbol{B}_i$$

$$\boldsymbol{V}_i^* = \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{D}^* \widetilde{\boldsymbol{Z}}'_{ci}$$

$$\boldsymbol{H}_i^* = \boldsymbol{B}_i^* \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{B}_i^* \widetilde{\boldsymbol{Z}}'_{bi} (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^* \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{\Sigma}_i^{-1}$$

$$\boldsymbol{E}_i^* = -\boldsymbol{E}_i \Big[-\boldsymbol{V}_i^{-1} \boldsymbol{V}_i^* \boldsymbol{V}_i^{-1} + \boldsymbol{H}^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i + \boldsymbol{H}'_i \Big(-\widetilde{\boldsymbol{Z}}'_{bi} (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^* \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{bi} \Big) \boldsymbol{H}_i +$$
$$\boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{H}_i^{*'} \Big] \boldsymbol{E}_i$$

$$\boldsymbol{T}_i^* = -\boldsymbol{T}_i \Bigg[\Big(\boldsymbol{E}_i^* \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \Big)^{b'} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^b -$$
$$(\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i^*)^{bb} (\boldsymbol{E}_i^{bb})^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^b +$$
$$(\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}_i^{bb})^{-1} \Big(\boldsymbol{E}_i^* \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \Big)^{b'} -$$
$$(\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) - \big(\boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \boldsymbol{H}'_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \big)' \boldsymbol{E}_i (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1}) - (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i^* (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1}) -$$
$$(\boldsymbol{H}'_i \boldsymbol{B}_i^{-1})' \boldsymbol{E}_i^* \big(\boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} - \boldsymbol{H}'_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) \big) \Bigg] \boldsymbol{T}_i.$$

Finally, the derivative of the expected value with respect to the residual variance of one of the continuous responses c_1, $\sigma^2_{c_1}$ equals

$$
\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} = \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}}, \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \sigma^2_{c1}} &= (\boldsymbol{E}^*_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{E}_i \boldsymbol{V}^{-1}_i \boldsymbol{S}^*_c \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^a + \\
&\quad \big[(\boldsymbol{E}^*_i)^{ab}(\boldsymbol{E}^{bb}_i)^{-1} - \boldsymbol{E}^{ab}_i((\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}^*_i)^{bb}(\boldsymbol{E}^{bb}_i)^{-1})\big] \\
&\quad \big[(\widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{ci}} - (\boldsymbol{E}_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})^b)\big] \\
&+ \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}\big[- \boldsymbol{E}^*_i \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{E}_i \boldsymbol{V}^{-1}_i \boldsymbol{S}^*_c \boldsymbol{V}^{-1}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\big]^b \\
&+ \Bigg((\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} - \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}^{-1}_i \boldsymbol{B}^*_i \boldsymbol{B}^{-1}_i)^a \\
&- \big[(\boldsymbol{E}^*_i)^{ab}(\boldsymbol{E}^{bb}_i)^{-1} - \boldsymbol{E}^{ab}_i((\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}^*_i)^{bb}(\boldsymbol{E}^{bb}_i)^{-1})\big](\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^b \\
&- \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_{\mathbf{i}}{}^{-1} - \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}^{-1}_i \boldsymbol{B}^*_i \boldsymbol{B}^{-1}_i)^b \Bigg) \\
&\times \Bigg(\boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad \ldots \quad -F_p(o_p) \big] + \boldsymbol{F}_i \Bigg) \\
&+ \Bigg((\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^a - \boldsymbol{E}^{ab}_i(\boldsymbol{E}^{bb}_i)^{-1}(\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_{\mathbf{i}}{}^{-1})^b \Bigg) \boldsymbol{tr}^*.
\end{aligned}
$$

To allow for a convenient solution for a general case, the following expression was evaluated numerically $\boldsymbol{tr}^* = \frac{\partial \boldsymbol{T}_i \begin{bmatrix} -F_1(o_1) & -F_2(o_2) & \dots & -F_p(o_p) \end{bmatrix} + \boldsymbol{F}_i}{\partial \sigma^2_{c_1}}$. In addition,

$$\begin{aligned}
\boldsymbol{S}^*_c &= \frac{\partial \boldsymbol{\Sigma}_i}{\partial \sigma^2_{c_1}} \\
\boldsymbol{K}^*_i &= \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}^*_c \boldsymbol{\Sigma}_i^{-1} \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{K}_i \\
\boldsymbol{B}^*_i &= \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}^*_i \widetilde{\boldsymbol{Z}}'_{bi} \boldsymbol{B}_i \\
\boldsymbol{H}^*_i &= \boldsymbol{B}^*_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}'_{bi} \boldsymbol{K}^*_i \widetilde{\boldsymbol{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}'_{ci} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}^*_c \boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}^*_i &= -\boldsymbol{E}_i \Big[-\boldsymbol{V}_i^{-1} \boldsymbol{S}^*_c \boldsymbol{V}_i^{-1} + \boldsymbol{H}^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i - \boldsymbol{H}' \boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1} \boldsymbol{H}_i + \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{H}^*_i \Big] \boldsymbol{E}_i \\
\boldsymbol{T}^*_i &= -\boldsymbol{T}_i \Bigg[\Big(\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \Big)^{b'} (\boldsymbol{E}^{bb}_i)^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^b - \\
&\quad (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}^{bb}_i)^{-1} (\boldsymbol{E}^*_i)^{bb} (\boldsymbol{E}^{bb}_i)^{-1} (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^b + \\
&\quad (\boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1})^{b'} (\boldsymbol{E}^{bb}_i)^{-1} \Big(\boldsymbol{E}^*_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} + \boldsymbol{E}_i \boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} - \boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \Big)^b - \\
&\quad (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) - \big(\boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} - \boldsymbol{H}'_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \big)' \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} - (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1})' \boldsymbol{E}^*_i (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1}) - \\
&\quad (\boldsymbol{H}'_i \boldsymbol{B}_i^{-1})' \boldsymbol{E}^*_i \big(\boldsymbol{H}^{*'}_i \boldsymbol{B}_i^{-1} \boldsymbol{H}'_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}^*_i \boldsymbol{B}_i^{-1}) \big) \Bigg] \boldsymbol{T}_i
\end{aligned}$$

D. Conditional distribution of continuous response(s) given the ordinal response(s)

D.1 *Expected value*

Let us derive (3.8), the conditional expected value of the $\widetilde{n}_i$-dimensional continuous subvector $\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}$ given the $\widetilde{p}_i$-dimensional ordinal subvector $\widetilde{\boldsymbol{Y}}_{\boldsymbol{bi}}$. This is equal to the integral over $\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}$ multiplied by the conditional distribution, which is defined as the quotient of (3.5) and (3.4).

$$
\begin{aligned}
&E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}_{\boldsymbol{2ik}} \leqslant \boldsymbol{c}] \\
&= \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=\infty} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \frac{\phi(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta};\boldsymbol{V}_i)\Phi(\boldsymbol{\gamma_c}-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}-\boldsymbol{\alpha}_i;\boldsymbol{B}_i)}{\Phi(\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta};\boldsymbol{L}_i^{-1})} d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&= c\int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=\infty} \int_{\boldsymbol{t}=-\infty}^{\boldsymbol{t}=\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}-\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(\widetilde{n}_i+\widetilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&\quad \times \exp\left\{-\frac{1}{2}\left((\widetilde{\boldsymbol{y}}_{ci}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{y}}_{ci}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})+(\boldsymbol{t}'\boldsymbol{B}_i^{-1}\boldsymbol{t})\right)\right\} d\boldsymbol{t}\ \mathrm{d}\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&= c\int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=\infty} \int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(\widetilde{n}_i+\widetilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&\quad \times \exp\left\{-\frac{1}{2}\left((\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})+(\boldsymbol{s}-\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}})'\boldsymbol{B}_i^{-1}(\boldsymbol{s}-\boldsymbol{H}_i\boldsymbol{y}_{ci})\right)\right\} d\boldsymbol{s} d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&= c\int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=\infty} \int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(\widetilde{n}_i+\widetilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \exp \\
&\quad \left\{-\frac{1}{2}\left((\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\boldsymbol{U}_i)'\boldsymbol{E}_i^{-1}(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\boldsymbol{U}_i)+O_i\right)\right\} d\boldsymbol{s} d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}},
\end{aligned}
$$

where

$$
\boldsymbol{H}_i = \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}_{ci}'\boldsymbol{\Sigma}_i^{-1}
$$
$$
c = \frac{1}{\Phi(\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta};\boldsymbol{L}_i^{-1})}.
$$

In addition, we have substituted $= -\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}$, and further

$$
(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})+(\boldsymbol{s}-\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}})'\boldsymbol{B}_i^{-1}(\boldsymbol{s}-\boldsymbol{H}_i\boldsymbol{y}_{ci}) = (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\boldsymbol{U}_i)'\boldsymbol{E}_i^{-1}(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}-\boldsymbol{U}_i)+O_i, \tag{D.1}
$$

where

$$\begin{aligned}
\boldsymbol{E}_i^{-1} &= \boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{V}_i^{-1}\\
\boldsymbol{l}_i' &= -\boldsymbol{s}'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\\
O_i &= \boldsymbol{s}'\boldsymbol{B}_i^{-1}\boldsymbol{s} + (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - (-\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{s} - \boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(-\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{s} - \boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\\
\boldsymbol{U}_i &= -\boldsymbol{E}_i l_i.
\end{aligned}$$

Integration over $\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}$ produces

$$\begin{aligned}
&E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}_{\boldsymbol{2ik}} \leqslant \boldsymbol{c}] \qquad \text{(D.2)}\\
&= c\int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{s} + \boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\exp\left\{-\frac{1}{2}O_i\right\}d\boldsymbol{s}\\
&= c\int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{s} + \boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\\
&\quad \exp\left\{-\frac{1}{2}\left((\boldsymbol{s}-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(\boldsymbol{s}-\boldsymbol{F}_i) + Gi\right)\right\}d\boldsymbol{s}\\
&= \int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{c}{(2\pi)^{\frac{\tilde{p}_i}{2}}}\frac{\sqrt{\boldsymbol{E}_i}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{s}\exp\left\{-\frac{1}{2}\left((\boldsymbol{s}-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(\boldsymbol{s}-\boldsymbol{F}_i) + Gi\right)\right\}d\boldsymbol{s}\\
&\quad + c\ \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\int_{\boldsymbol{s}=-\infty}^{\boldsymbol{s}=\boldsymbol{\gamma}_c-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta}+\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\tilde{p}_i}{2}}}\exp\left\{-\frac{1}{2}\left((\boldsymbol{s}-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(\boldsymbol{s}-\boldsymbol{F}_i) + Gi\right)\right\}d\boldsymbol{s},
\end{aligned}$$

where we have subsituted $O_i = (\boldsymbol{s}-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(\boldsymbol{s}-\boldsymbol{F}_i) + G_i$,

with

$$\begin{aligned}
\boldsymbol{T}_i^{-1} &= \boldsymbol{B}_i^{-1} - (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})\\
\boldsymbol{F}_i &= \boldsymbol{T}_i \cdot (\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\\
G_i &= -\boldsymbol{F}_i'\boldsymbol{T}_i^{-1}\boldsymbol{F}_i + (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) - (\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}).
\end{aligned}$$

Now consider,

$$
\begin{aligned}
\boldsymbol{T}_i &= \boldsymbol{B}_i - \boldsymbol{H}_i\big(- \boldsymbol{E}_i^{-1} + \boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\big)^{-1}\boldsymbol{H}_i \\
&= \boldsymbol{B}_i - \boldsymbol{H}_i\big(- \boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i - \boldsymbol{V}_i^{-1} + \boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\big)^{-1}\boldsymbol{H}_i \\
&= \boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&\boldsymbol{F}_i'\boldsymbol{T}_i^{-1}\boldsymbol{F}_i \\
&= \Big[(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\Big]'\boldsymbol{T}_i\Big[(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\Big] \\
&= \Big[(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\Big]'(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i')\Big[(\boldsymbol{H}_i'\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})\Big] \\
&= (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&\quad +(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\
&= (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i(\boldsymbol{E}_i^{-1} - \boldsymbol{V}_i^{-1})\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&\quad +(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i(\boldsymbol{E}_i^{-1} - \boldsymbol{V}_i^{-1})\boldsymbol{V}_i(\boldsymbol{E}_i^{-1} - \boldsymbol{V}_i^{-1})\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\
&= (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&\quad +(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\
&= -(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + (\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}
\end{aligned}
$$

Hence, $G_i = 0$.

The first term can be solved by calculating the expected value of the truncated normal distribution, as described by Manjunath and Wilhelm (2021).

$$
\begin{aligned}
&E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}_{\boldsymbol{2ik}} \leqslant \boldsymbol{c}] \\
&= c\sqrt{\frac{|\boldsymbol{E}_i||\boldsymbol{T}_i|}{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{T}_i\Phi(o,\boldsymbol{T}_i)\big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big] \\
&\quad +c\sqrt{\frac{|\boldsymbol{E}_i||\boldsymbol{T}_i|}{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i, \boldsymbol{T}_i)\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{F}_i \\
&\quad +c\sqrt{\frac{|\boldsymbol{E}_i||\boldsymbol{T}_i|}{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i, \boldsymbol{T}_i)\boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \\
&= c\sqrt{\frac{|\boldsymbol{E}_i||\boldsymbol{T}_i|}{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i, \boldsymbol{T}_i)\Bigg(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{F}_i + \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \\
&\quad \boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{T}_i\Bigg(\big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big]\Bigg)\Bigg),
\end{aligned}
$$

where o, $F_i(x_i)$ and $\varphi(x_i)$ are defined in (C.3).

Now, consider

$$
\begin{aligned}
\frac{|\boldsymbol{E}_i||\boldsymbol{T}_i|}{|\boldsymbol{V}_i||\boldsymbol{B}_i|} &= \frac{|\boldsymbol{B}_i^{-1}||\boldsymbol{T}_i|}{|\boldsymbol{V}_i||\boldsymbol{E}_i^{-1}|} \\
&= \frac{|\boldsymbol{B}_i^{-1}||\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'|}{|\boldsymbol{V}_i||\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{V}_i^{-1}|} \\
&= \frac{|\boldsymbol{I}_p + \boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}_i'|}{|\boldsymbol{V}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i + \boldsymbol{I}_n|} \\
&= 1,
\end{aligned} \tag{D.3}
$$

where we used a familiar form of the Sylvester's identity: det(I+AB)=det(I+BA).

Next, consider

$$
\begin{aligned}
\boldsymbol{T}_i &= \boldsymbol{B}_i + \boldsymbol{H}_i \boldsymbol{V}_i \boldsymbol{H}_i' \\
&= \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{ci}' \boldsymbol{\Sigma}_i^{-1} (\boldsymbol{\Sigma}_i + \boldsymbol{Z}_{ci} \boldsymbol{D}_i \boldsymbol{Z}_{ci}') \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{ci} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i \\
&= \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{ci}' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{ci} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{ci}' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{ci} \boldsymbol{D}_i \boldsymbol{Z}_{ci}' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{ci} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i \\
&= \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i (\boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi}) \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i (\boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi}) \boldsymbol{D}_i (\boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi}) \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i \\
&= \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{D}_i^{-1} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{D}_i (\boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi}) \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i (\boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi}) \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{D}_i (\boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi}) \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i \\
&= \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{D}_i^{-1} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{D}_i^{-1} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i \\
&= \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i - \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i + \\
&\quad \boldsymbol{B}_i \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{D}_i \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi} \boldsymbol{K}_i \boldsymbol{Z}_{bi}' \boldsymbol{B}_i,
\end{aligned}
$$

where we subsituted $\boldsymbol{Z}_{ci}' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{ci} = \boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}_{bi}' \boldsymbol{Z}_{bi}$.

Next,

$$\begin{aligned}
\boldsymbol{T}_i &= \boldsymbol{B}_i + \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i - \boldsymbol{B}_i(-\boldsymbol{B}_i^{-1}+\boldsymbol{I}_p)\boldsymbol{B}_i - \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}(-\boldsymbol{B}_i^{-1}+\boldsymbol{I}_p)\boldsymbol{B}_i - \\
&\quad \boldsymbol{B}_i(-\boldsymbol{B}_i^{-1}+\boldsymbol{I}_p)\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i + \boldsymbol{B}_i(-\boldsymbol{B}_i^{-1}+\boldsymbol{I}_p)(-\boldsymbol{B}_i^{-1}+\boldsymbol{I}_p)\boldsymbol{B}_i + \\
&\quad \boldsymbol{B}_i(-\boldsymbol{B}_i^{-1}+\boldsymbol{I}_p)\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}(-\boldsymbol{B}_i^{-1}+\boldsymbol{I}_p)\boldsymbol{B}_i \\
&= \boldsymbol{B}_i + \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i + \boldsymbol{B}_i - \boldsymbol{B}_i\boldsymbol{B}_i + \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi} - \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i - \\
&\quad +\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i - \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i + \boldsymbol{I}_p - \boldsymbol{B}_i - \boldsymbol{B}_i + \boldsymbol{B}_i\boldsymbol{B}_i + \\
&\quad \boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi} - \boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i - \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi} + \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi}\boldsymbol{B}_i \\
&= \boldsymbol{I}_p + \boldsymbol{Z}_{bi}\boldsymbol{D}_i\boldsymbol{Z}'_{bi} \\
&= \boldsymbol{L}_i^{-1},
\end{aligned} \tag{D.4}$$

where we rewrite $\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{bi} = -\boldsymbol{B}_i^{-1} + \boldsymbol{I}_p$.

Now, consider

$$\begin{aligned}
\boldsymbol{F}_i &= \boldsymbol{T}_i \cdot (\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})'\boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) \\
&= \boldsymbol{T}_i(\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})(\boldsymbol{V}_i - \boldsymbol{V}_i\boldsymbol{H}'_i(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i)^{-1})\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= (\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i)(\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})'\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad (\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i)(\boldsymbol{H}'_i\boldsymbol{B}_i^{-1})'\boldsymbol{V}_i\boldsymbol{H}'_i(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i)^{-1})\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \\
&= \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + (\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{V}_i(\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}))((\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1})'\boldsymbol{B}_i^{-1})'\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad \boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{V}_i(\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1})(\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i)^{-1}\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \\
&\quad (\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{V}_i(\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}))((\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1})'\boldsymbol{B}_i^{-1})'\boldsymbol{V}_i(\boldsymbol{B}_i\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1})' \\
&\quad (\boldsymbol{B}_i + \boldsymbol{H}_i\boldsymbol{V}_i\boldsymbol{H}'_i)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}
\end{aligned}$$

After substituting $\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Z}_{ci} = \boldsymbol{K}_i^{-1} - \boldsymbol{D}_i^{-1} - \boldsymbol{Z}'_{bi}\boldsymbol{Z}_{bi}$, $\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{bi} = -\boldsymbol{B}_i^{-1} + \boldsymbol{I}_p$ and eliminating terms this becomes

$$\begin{aligned}
\boldsymbol{F}_i &= (\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi} + \boldsymbol{I}_p)\boldsymbol{Z}_{bi}\boldsymbol{K}_i\boldsymbol{Z}'_{ci}\boldsymbol{\Sigma}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} \qquad \text{(D.5)}\\
&\quad +(\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi} + \boldsymbol{I}_p)(\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi} + \boldsymbol{I}_p)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&\quad -(\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi} + \boldsymbol{I}_p)\boldsymbol{B}_i^{-1}(\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi} + \boldsymbol{I}_p)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&\quad -(\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi} + \boldsymbol{I}_p)\boldsymbol{B}_i^{-1}\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi}(\boldsymbol{Z}_{bi}\boldsymbol{D}\boldsymbol{Z}'_{bi} + \boldsymbol{I}_p)^{-1}\boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\\
&= \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}
\end{aligned}$$

Combing the results of (D.3),(D.4),(D.5), the expected value simplifies to

$$\begin{aligned}
&E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{Y}}_{\boldsymbol{2ik}} \leqslant \boldsymbol{c}]\\
&= \boldsymbol{E}_i\Big(\boldsymbol{H}'_i\boldsymbol{B}_i^{-1}\boldsymbol{F}_i + \boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{H}'_i\boldsymbol{B}_i^{-1}\boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad \ldots \quad -F_p(o_p)\big]\Big)
\end{aligned}$$

D.2 *Prediction interval*

The prediction interval of the expected values of a (sub)vector of the continuous response given a (sub)vector of the ordinal response is composed by the variability of the observations (second central moment) and the standard errors of the transformed parameters. We will first derive the second central moment and then derive the standard errors via the delta method.

The uncertainty of a new observation is defined as the second central moment

$$
\begin{aligned}
&E\Big[\big(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)\big(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)'\Big] \\
&= \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=+\infty} (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})](\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}])' \cdot f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&= \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=+\infty} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}{}' f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) + E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})' f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) \\
&\quad -(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})' + E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}{}') \cdot f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&= \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=+\infty} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}{}' f(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) d\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})' \\
&= c \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=+\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}'_{\boldsymbol{ci}} \\
&\quad \exp\bigg\{ -\frac{1}{2}\bigg((\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})' \boldsymbol{V}_i^{-1} (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + ({}'\boldsymbol{B}_i^{-1}) \bigg) \bigg\} dd\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&\quad -E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})' \\
&= c \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=+\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}'_{\boldsymbol{ci}} \\
&\quad \exp\bigg\{ -\frac{1}{2}\bigg((\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})' \boldsymbol{V}_i^{-1} (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + (-\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}{}')\boldsymbol{B}_i^{-1}(-\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}) \bigg) \bigg\} dd\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} \\
&\quad -E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})' \\
&= c \int_{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=-\infty}^{\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}=+\infty} \int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{(\tilde{n}_i+\tilde{p}_i)}{2}}} \frac{1}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}} \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}\widetilde{\boldsymbol{y}}'_{\boldsymbol{ci}} \\
&\quad \exp\bigg\{ -\frac{1}{2}\bigg((\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \boldsymbol{u}_i)' \boldsymbol{E}_i^{-1} (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \boldsymbol{u}_i) + O_i \bigg) \bigg\} dd\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}) E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})'
\end{aligned}
$$

where we have substituted $= -\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}$, and further

$$
(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})' \boldsymbol{V}_i^{-1} (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}) + (-\boldsymbol{H}_i\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}})' \boldsymbol{B}_i^{-1} (-\boldsymbol{H}_i\boldsymbol{y}_{ci}) = (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \boldsymbol{u}_i)' \boldsymbol{E}_i^{-1} (\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \boldsymbol{u}_i) + O_i,
$$

where $\boldsymbol{E}_i^{-1}$,O_i and $\boldsymbol{u}_i$ are defined in (D.1).

Integrating over $\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}$ yields the following result

$$
\begin{aligned}
&E\bigg[\big(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)\big(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)'\bigg] \\
&= c\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\widetilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\boldsymbol{u}_i\boldsymbol{u}_i'\exp\bigg\{-\frac{1}{2}O_i\bigg\}d \\
&\quad + \boldsymbol{E}_i c\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\widetilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\exp\bigg\{-\frac{1}{2}O_i\bigg\}d \\
&\quad - E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})' \\
&= c\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\widetilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\boldsymbol{u}_i\boldsymbol{u}_i'\exp\bigg\{-\frac{1}{2}\Big((-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(-\boldsymbol{F}_i) + G_i\Big)\bigg\}d \\
&\quad + \boldsymbol{E}_i c\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}} \frac{1}{(2\pi)^{\frac{\widetilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\exp\bigg\{-\frac{1}{2}\Big((-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(-\boldsymbol{F}_i) + G_i\Big)\bigg\}d \\
&\quad - E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})',
\end{aligned}
$$

where we have rewritten $O_i = (-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(-\boldsymbol{F}_i) + G_i$. In (D.2), $\boldsymbol{F}_i$, $\boldsymbol{T}_i$ and G_i are defined.

Integration over yields the following result

$$
\begin{aligned}
&E\bigg[\big(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)\big(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} - E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]\big)'\bigg] \\
&= c\frac{1}{(2\pi)^{\frac{\widetilde{p}_i}{2}}}\frac{\sqrt{|\boldsymbol{E}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\int_{=-\infty}^{=\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}}\Big(\boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1\prime}\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i \\
&\quad + \boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i \\
&\quad + \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i + \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i)\Big) \\
&\quad \exp\bigg\{-\frac{1}{2}\Big(\big(-\boldsymbol{F}_i)'\boldsymbol{T}_i^{-1}(-\boldsymbol{F}_i)\big)\Big)\bigg\}d \\
&\quad + c\boldsymbol{E}_i\frac{\sqrt{|\boldsymbol{E}_i||\boldsymbol{T}_i|}}{\sqrt{|\boldsymbol{V}_i||\boldsymbol{B}_i|}}\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i) - E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})' \\
&= \boldsymbol{E}_i + \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i + \boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\Big(\boldsymbol{N} + \boldsymbol{J}\boldsymbol{F}_i' + \boldsymbol{F}_i\boldsymbol{J} + \boldsymbol{F}_i\boldsymbol{F}_i'\Big)\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i + \\
&\quad \boldsymbol{E}_i\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}\boldsymbol{J}'\boldsymbol{B}_i^{-1}\boldsymbol{H}_i\boldsymbol{E}_i + \boldsymbol{E}_i\boldsymbol{H}_i'\boldsymbol{B}_i^{-1}\boldsymbol{J}(\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta})'\boldsymbol{V}_i^{-1}\boldsymbol{E}_i - E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})E(\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c})',
\end{aligned}
$$

where $\boldsymbol{J}$ is the expected value of the truncated multivariate normal density, and $\boldsymbol{N}$ is the second central moment of the latter density. These are defined in C.13.

Agresti (2002) derived the distribution of transformed maximum likelihood parameters via the delta method

$$G(\hat{\boldsymbol{\theta}}) \to N\left(\boldsymbol{\theta}, \left(\frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)' Var(\hat{\boldsymbol{\theta}}) \frac{\partial G(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right),$$

where $\boldsymbol{\theta}$ denotes the parameter vector. We will first sketch the derivative of the expected value with respect to a coefficient β_{c2} of a predictor $\widetilde{\boldsymbol{X}}_{c2}$ of the continuous response and next sketch the derivation of a coefficient β_{b2} of a predictor $\widetilde{\boldsymbol{X}}_{b2}$ of the ordinal response. Next, the gradients of the variance parameters will be derived. The gradient of a coefficient β_{c2} of a predictor $\widetilde{\boldsymbol{X}}_{c2}$ equals the following:

$$\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} | \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \beta_{c2}} = \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}'_{c2i} + \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \boldsymbol{T}_i \boldsymbol{B}_i^{-1} \boldsymbol{H}_i \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}'_{c2i} + \boldsymbol{E}_i \boldsymbol{H}'_i \boldsymbol{B}_i^{-1} \nu,$$

with

$$\begin{aligned}
o &= \boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i \\
\lambda &= \sum_{k=1}^{\widetilde{p}_i} (\boldsymbol{H}_i \widetilde{\boldsymbol{X}}'_{c2i} - \boldsymbol{T}_i \boldsymbol{B}_i^{-1} \boldsymbol{H}_i \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{c2i})_k \phi((\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, T_{i,kk}) \\
&\quad \times \Phi((\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}'_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}, \boldsymbol{T}_{i,-k|k}), \\
\nu &= \frac{\sum_{k=1}^{\widetilde{p}_i} (\boldsymbol{H}_i \widetilde{\boldsymbol{X}}'_{c2i} - \boldsymbol{T}_i \cdot \boldsymbol{B}_i^{-1} \boldsymbol{H}_i \boldsymbol{E}_i \boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{c2i})_k g_k(o_k) - \lambda \boldsymbol{T}_i \big[- F_1(o_1) \quad - F_2(o_2) \quad \ldots \quad - F_p(o_p) \big]}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i, \boldsymbol{T}_i)} \\
g_k(x_k) &= \int_{-\infty}^{o_1} \cdots \int_{-\infty}^{o_{i-1}} \int_{-\infty}^{o_{i+1}} \cdots \int_{-\infty}^{o_{\widetilde{p}_i}} [x_1..x_{k-1} o_k x_{k+1}..x_{\widetilde{p}_i}]' \varphi([x_1..x_{k-1} o_k x_{k+1}..x_{\widetilde{p}_i}]', T_i) dx_{-k}, \\
\varphi(x) &= \begin{cases} \frac{\phi(x, \boldsymbol{T}_i)}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{T}_i)}, & \text{for } \boldsymbol{x} \leqslant \boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

and $\boldsymbol{T}_{i,-k|k}$ is defined in C.14.

Next, for a coefficient β_{b2} of a predictor $\widetilde{\boldsymbol{X}}_{b2}$ of the ordinal response the derivative is the following

$$\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \beta_{b2}} = \boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \frac{-\sum_{k=1}^{\widetilde{p}_i} \widetilde{\boldsymbol{X}}'_{b2ik} g_k(o_k) - \Omega \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i)},$$

where

$$\Omega = -\sum_{k=1}^{\widetilde{p}} \widetilde{\boldsymbol{X}}'_{b2ik} \phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, \boldsymbol{T}_{i,kk}\big] \Phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}'_{b2i}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}; \boldsymbol{T}_{i,-k|k}\big].$$

Further, the derivative with respect to the threshold value γ_c equals

$$\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \gamma_c} = \boldsymbol{E}_i \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \frac{-\sum_{k=1}^{\widetilde{p}_i} \widetilde{\boldsymbol{X}}'_{b2ik} g_k(o_k) - \omega \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]}{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}, \boldsymbol{F}_i, \boldsymbol{T}_i)},$$

where

$$\omega = \sum_{k=1}^{\widetilde{p}} \phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_k, \boldsymbol{T}_{i,kk}\big] \Phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}'_{b2i}\boldsymbol{\beta} + \boldsymbol{H}_i \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} - \boldsymbol{F}_i)_{-k}; \boldsymbol{T}_{i,-k|k}\big].$$

The derivative of the expected value with respect to an arbitrary component of $\boldsymbol{D}$, denoted by τ equals

$$\begin{aligned}\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}}|\widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \tau} = \quad & \boldsymbol{E}_i^*(\boldsymbol{V}_i^{-1}\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{H}'\boldsymbol{B}_i^{-1}(\boldsymbol{F}_i + \boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big])) - \\ & \boldsymbol{E}_i(\boldsymbol{V}_i^{-1}\boldsymbol{V}_i^*\boldsymbol{V}_i^{-1})\widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta} + \boldsymbol{E}_i\boldsymbol{H}_i^{*'}\boldsymbol{B}_i^{-1}(\boldsymbol{F}_i + \boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big]) \\ & -\boldsymbol{E}_i\boldsymbol{H}_i(\boldsymbol{B}_i^{-1}\boldsymbol{B}_i^*\boldsymbol{B}_i^{-1})(\boldsymbol{F}_i + \boldsymbol{T}_i\big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big]) + \\ & \boldsymbol{E}_i\boldsymbol{H}_i\boldsymbol{B}_i^{-1}\boldsymbol{tr}^*\end{aligned}$$

To allow for a convenient solution for a general case, the following expressions were evaluated numerically

$$\boldsymbol{tr}^* = \frac{\partial \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p)\big] + \boldsymbol{F}_i}{\partial \tau}$$

In addition,

$$\begin{aligned}
\boldsymbol{D}^* &= \frac{\partial \boldsymbol{D}}{\partial \tau} \\
\boldsymbol{B}_i^* &= \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi}\ (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^* \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{bi}^{'} \boldsymbol{B}_i \\
\boldsymbol{V}_i^* &= \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{D}^* \widetilde{\boldsymbol{Z}}_{ci}^{'} \\
\boldsymbol{H}_i^* &= \boldsymbol{B}_i^* \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^* \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}_i^* &= -\boldsymbol{E}_i \Big[-\boldsymbol{V}_i^{-1} \boldsymbol{V}_i^* \boldsymbol{V}_i^{-1} + \boldsymbol{H}^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i + \boldsymbol{H}_i' \Big(-\widetilde{\boldsymbol{Z}}_{bi}\ (\boldsymbol{K}_i \boldsymbol{D}^{-1} \boldsymbol{D}^* \boldsymbol{D}^{-1} \boldsymbol{K}_i) \widetilde{\boldsymbol{Z}}_{bi}^{'} \Big) \boldsymbol{H}_i + \\
&\qquad \boldsymbol{H}_i' \boldsymbol{B}_i^{-1} \boldsymbol{H}_i^* \Big] \boldsymbol{E}_i
\end{aligned}$$

Lastly, the derivative of the expected value with respect to $\sigma_{c_1}^2$, the residual variance of continuous response c_1, equals

$$\begin{aligned}
\frac{\partial E[\widetilde{\boldsymbol{Y}}_{\boldsymbol{ci}} | \widetilde{\boldsymbol{y}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}]}{\partial \sigma_{c_1}^2} = \quad & \boldsymbol{E}_i^* (\boldsymbol{V}_i^{-1} \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \boldsymbol{H}' \boldsymbol{B}_i^{-1} (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big])) - \\
& \boldsymbol{E}_i (\boldsymbol{V}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{V}_i^{-1}) \widetilde{\boldsymbol{X}}_{ci} \boldsymbol{\beta} + \boldsymbol{E}_i \boldsymbol{H}_i^{*'} \boldsymbol{B}_i^{-1} (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) \\
& -\boldsymbol{E}_i \boldsymbol{H}_i (\boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1}) (\boldsymbol{F}_i + \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big]) + \\
& \boldsymbol{E}_i \boldsymbol{H}_i \boldsymbol{B}_i^{-1} \boldsymbol{tr}^*
\end{aligned}$$

To allow for a convenient solution for a general case, the following expressions were evaluated numerically

$$\boldsymbol{tr}^* = \frac{\partial \boldsymbol{T}_i \big[-F_1(o_1) \quad -F_2(o_2) \quad ... \quad -F_p(o_p) \big] + \boldsymbol{F}_i}{\partial \sigma_{c_1}^2}$$

In addition,

$$\begin{aligned}
\boldsymbol{S}_c^* &= \frac{\partial \boldsymbol{\Sigma}_i}{\partial \sigma_{c_1}^2} \\
\boldsymbol{K}_i^* &= \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{\Sigma}_i^{-1} \widetilde{\boldsymbol{Z}}_{ci} \boldsymbol{K}_i \\
\boldsymbol{B}_i^* &= \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i^* \widetilde{\boldsymbol{Z}}_{bi}^{'} \boldsymbol{B}_i \\
\boldsymbol{H}_i^* &= \boldsymbol{B}_i^* \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} + \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i^* \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{B}_i \widetilde{\boldsymbol{Z}}_{bi} \boldsymbol{K}_i \widetilde{\boldsymbol{Z}}_{ci}^{'} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{\Sigma}_i^{-1} \\
\boldsymbol{E}_i^* &= -\boldsymbol{E}_i \left[-\boldsymbol{V}_i^{-1} \boldsymbol{S}_c^* \boldsymbol{V}_i^{-1} + \boldsymbol{H}^{*'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i - \boldsymbol{H}^{'} \boldsymbol{B}_i^{-1} \boldsymbol{B}_i^* \boldsymbol{B}_i^{-1} \boldsymbol{H}_i + \boldsymbol{H}_i^{'} \boldsymbol{B}_i^{-1} \boldsymbol{H}_i^* \right] \boldsymbol{E}_i
\end{aligned}$$

E. Standard errors of the conditional probability of the ordinal response conditional on the continuous response

The conditional probability can be expressed as follows:

$$f(\widetilde{\boldsymbol{y}}^{\boldsymbol{a}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c} | \widetilde{\boldsymbol{y}}^{\boldsymbol{b}}_{\boldsymbol{bi}} \leqslant \boldsymbol{c}, \widetilde{\boldsymbol{y}}_{\boldsymbol{ci}}) = \frac{\Phi(\gamma_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_{\mathbf{i}})}{\Phi(\boldsymbol{\gamma}^b_c - \widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)},$$

Once the logit transformation is applied to confine the boundaries within the unit interval, the confidence interval can be determined using the delta method. Let us assume that z represents the logit transformation of the conditional probability (3.9). The gradient of a coefficient β_{c2} related to a predictor of a continuous response $\mathbf{X}_{c2}$ can be expressed as follows:

$$\begin{aligned}\frac{\partial z}{\partial \beta_{c2}} = \Bigg\{ &\sum_{k=1}^{\widetilde{p}_i} \boldsymbol{H}_{ik}\widetilde{\boldsymbol{X}}_{c2i}\phi\big[(\gamma_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}\big]\Phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}\big] \\ &\Phi(\boldsymbol{\gamma}^b_c - \widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i) \\ -&\sum_{k=1}^{\widetilde{p}^b_i} \boldsymbol{H}^b_{ik}\boldsymbol{X}_{c2i}\phi\big[(\boldsymbol{\gamma}^b_c - \widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B^{bb}_{kk}\big] \\ &\Phi\big[(\boldsymbol{\gamma}^b_c - \widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_{-k}; \mathbf{B}^{bb}_{-k|k}\big] \\ &\Phi\big[\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i\big]\Bigg\} \\ &\frac{-\left(\Phi\big[\boldsymbol{\gamma}^b_c - \widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i\big]\right)^{-2}}{\left(\frac{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\boldsymbol{\gamma}^b_c - \widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)}\right)^2 - \left(\frac{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\boldsymbol{\gamma}^b_c - \widetilde{\boldsymbol{X}}^b_{bi}\boldsymbol{\beta} - \boldsymbol{H}^b_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}^{bb}_i)}\right)},\end{aligned}$$

In addition, $\mathbf{B}_i$ is partitioned as follows

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{B}^{(k)}_{11} & \mathbf{B}^{(k)}_{12} \\ \mathbf{B}^{(k)}_{21} & B_{kk} \end{bmatrix}.$$

Next, $\mathbf{B}_{-k|k}$ is defined as

$$\mathbf{B}_{-k|k} = \mathbf{B}^{(k)}_{11} - \mathbf{B}^{(k)}_{12} B^{-1}_{kk} \mathbf{B}^{(k)}_{21}, \tag{E.1}$$

which has been retrieved from Poddar (2016), in their Appendix A.

Next, the gradient of a coefficient β_{b2} of one of the predictors of the ordinal responses $\widetilde{\boldsymbol{X}}_{b2}$ is defined as

$$\frac{\partial z}{\partial \beta_{b2}} = \frac{\left\{ \begin{array}{l} \sum_{k=1}^{\widetilde{p}_i} -X_{b2ik}\phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}\big]\Phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}\big] \\ \Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}) \\ + \sum_{k=1}^{\widetilde{p}_i^b} X_{b2ik}^b\phi\big[(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; \boldsymbol{B}_{kk}^{bb}\big] \\ \Phi\big[(\boldsymbol{\gamma}_c^b - \widetilde{X}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_{-k}; \mathbf{B}_{-k|k}^{bb}\big] \\ \Phi\big[\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i\big] \end{array} \right\} \\ -\left(\Phi\big[\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}\big]\right)^{-2}}{\left(\frac{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)^2 - \left(\frac{\Phi(-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)}.$$

Similarly, the gradient of the threshold value γ_c equals

$$\frac{\partial z}{\partial \gamma_c} = \frac{\left\{ \begin{array}{l} \sum_{k=1}^{\widetilde{p}_i} \phi\big[(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}\big]\Phi\big[(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}\big] \\ \times \Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}) \\ - \sum_{k=1}^{\widetilde{p}_i^b} \phi\big[(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_k; B_{kk}^{bb}\big]\Phi\big[(\boldsymbol{\gamma}_c^b - \widetilde{X}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}))_{-k}; \mathbf{B}_{-k|k}^{bb}\big] \\ \Phi\big[\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i\big] \end{array} \right\} \\ -\left(\Phi\big[\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}\big]\right)^{-2}}{\left(\frac{\Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)^2 - \left(\frac{\Phi(-\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)}. \tag{E.2}$$

To allow for a convenient solution for a general case, the following expressions were evaluated numerically

$$st^* = \frac{\partial \Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_{\mathbf{i}})}{\partial \sigma_{c_1}^2},$$

$$sn^* = \frac{\partial \Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}{\partial \sigma_{c_1}^2},$$

$$dt^* = \frac{\partial \Phi(\boldsymbol{\gamma}_c - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_{\mathbf{i}})}{\partial \tau},$$

$$dn^* = \frac{\partial \Phi(\boldsymbol{\gamma}_c^b - \widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}{\partial \tau}.$$

The gradients with respect to the residual error variance of response c_1, $\sigma_{c_1}^2$, and an arbitrary component of the variance-covariance matrix of the random effects τ, equal

$$\frac{\partial z}{\partial \sigma_{c_1}^2} = \frac{\begin{array}{l}\Big\{\gamma_c^b - st^*\Phi(\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}) - \\ sn^*\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)\Big\} \\ \qquad -\Big(\Phi\big[\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}\big]\Big)^{-2}\end{array}}{\left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)^2 - \left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)},$$

$$\frac{\partial z}{\partial \tau} = \frac{\begin{array}{l}\Big\{dt^*\Phi(\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}) - \\ dn^*\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)\Big\} \\ \qquad -\Big(\Phi\big[\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb}\big]\Big)^{-2}\end{array}}{\left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)^2 - \left(\frac{\Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{H}_i(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i)}{\Phi(\widetilde{\boldsymbol{X}}_{bi}^b\boldsymbol{\beta} - \boldsymbol{H}_i^b(\widetilde{\boldsymbol{y}}_{\boldsymbol{ci}} - \widetilde{\boldsymbol{X}}_{ci}\boldsymbol{\beta}); \boldsymbol{B}_i^{bb})}\right)}.$$

Next, the 95% confidence interval can be constructed as

$$\operatorname{expit}\left\{z \pm 1.96\sqrt{\left(\frac{\partial z}{\partial \boldsymbol{\theta}}\right)' \operatorname{Var}(\hat{\boldsymbol{\theta}}) \left(\frac{\partial z}{\partial \boldsymbol{\theta}}\right)}\right\}, \tag{E.3}$$

where $\boldsymbol{\theta}$ signals the parameter vector.

F. Standard errors of the conditional probability of the ordinal response conditional on the continuous response

The standard errors can be calculated via the delta method, in analogy with the standard errors of (3.9). Let z be the logit transformed conditional probability of (3.10). The derivative of z with respect to a coefficient β_{c2} of a predictor of the continuous response vector $\mathbf{X}_{c2}$ is defined as follows

$$\frac{\partial z}{\partial \beta_{c2}} = -\frac{\sum_{k=1}^{\widetilde{p}_i} \boldsymbol{H}_{ik}\boldsymbol{X}_{c2i}\phi\big[(\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_k; B_{kk}\big]\Phi\big[(\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \mathbf{B}_{-k|k}\big]}{\Big(\Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]\Big)^2 - \Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]},$$

where $\mathbf{B}_{-k|k}$ is defined in E.1.Next, the gradient of a coefficient β_{b2} of one of the predictors of the binary response vector X_{b2} is defined as

$$\frac{\partial z}{\partial \beta_{b2}} = \frac{\sum_{k=1}^{\widetilde{p}_i} X_{b2ik}\phi\big[(\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_k; B_{kk}\big]\Phi\big[(\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \boldsymbol{B}_{-k|k}\big]}{\Big(\Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]\Big)^2 - \Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]}.$$

The gradient of the threshold value γ_c equals

$$\frac{\partial z}{\partial \gamma_c} = -\frac{\sum_{k=1}^{\widetilde{p}_i} \phi\big[(\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_k; B_{kk}\big]\Phi\big[(\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i)_{-k}; \boldsymbol{B}_{-k|k}\big]}{\Big(\Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]\Big)^2 - \Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]}.$$

Next, the gradients with respect to the residual variance of a continuous response ,$\sigma^2_{c_1}$, and an arbitrary component of the variance covariance matrix τ will be derived. To allow for a convenient solution for a general case, the following expressions were evaluated numerically

$$s^* = \frac{\partial \Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \boldsymbol{B}_{\mathbf{i}}))}{\partial \sigma^2_{c_1}},$$
$$d^* = \frac{\partial \Phi(\widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \boldsymbol{B}_{\mathbf{i}})}{\partial \tau}.$$

The gradients can be expressed as follows

$$\frac{\partial z}{\partial \sigma_{c_1}^2} = \frac{-s^*}{\left(\Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]\right)^2 - \Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]},$$
$$\frac{\partial z}{\partial \tau} = \frac{-d^*}{\left(\Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]\right)^2 - \Phi\big[\boldsymbol{\gamma_c} - \widetilde{\boldsymbol{X}}_{bi}\boldsymbol{\beta} - \boldsymbol{\alpha}_i; \mathbf{B}_i\big]}$$

Next, the standard errors can be obtained from the gradients with (E.3).

G. Computation of the correlation function

We will sketch the derivation of the manifest correlations (3.11). The expected value of the product can be written as follows, using the independence of the elements of $\boldsymbol{Y}_{ci}$ and $\boldsymbol{Y}_{bi}$ conditional on the random effects q-dimensional vector of the random effects $\boldsymbol{\xi}_i$. The coefficients of the random effects are the $q \times 1$ vector $\boldsymbol{z}_{1ij}$ for the continuous response and the $q \times 1$ vector for the binary response $\boldsymbol{z}_{2ik}$. The derivation starts as follows:

$$
\begin{aligned}
&E[Y_{1ij}, Y_{2ik} \leqslant c] \\
&= E_{\boldsymbol{\xi}_i}[E(Y_{1ij}, Y_{2ik \leqslant c}|\boldsymbol{\xi}_i)] \\
&= E_{\boldsymbol{\xi}_i}[E(Y_{1ij}|\boldsymbol{\xi}_i)E(Y_{2ik} \leqslant c|\boldsymbol{\xi}_i)] \\
&= E_{\boldsymbol{\xi}_i}[(\boldsymbol{x}'_{1ij}\boldsymbol{\beta} + \boldsymbol{z}'_{1ij}\boldsymbol{\xi}_i) \cdot \Phi(\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ij}\boldsymbol{\beta} - \boldsymbol{z}'_{2ij}\boldsymbol{\xi}_i)] \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{t=-\infty}^{t=\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta} - \boldsymbol{z}'_{2ik}\boldsymbol{\xi}_i} \frac{1}{(2\pi)^{q/2}|\boldsymbol{D}|^{1/2}\sqrt{2\pi}} \exp\Big\{-\frac{1}{2}(\boldsymbol{\xi}'_i\boldsymbol{D}^{-1}\boldsymbol{\xi}_i + t^2)\Big\} \\
&\quad (\boldsymbol{x}'_{1ij}\boldsymbol{\beta} + \boldsymbol{z}'_{1ij}\boldsymbol{\xi}_i) d\boldsymbol{\xi}_i dt \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{s=-\infty}^{s=\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}} \frac{1}{(2\pi)^{q/2}|\boldsymbol{D}|^{1/2}\sqrt{2\pi}} \exp\Big\{-\frac{1}{2}(\boldsymbol{\xi}'_i\boldsymbol{D}^{-1}\boldsymbol{\xi}_i + (s - \boldsymbol{z}'_{2ik}\boldsymbol{\xi}_i)^2)\Big\} \\
&\quad (\boldsymbol{x}'_{1ij}\boldsymbol{\beta} + \boldsymbol{z}'_{1ij}\boldsymbol{\xi}_i) d\boldsymbol{\xi}_i ds \\
&= \Big(\int_{-\infty}^{+\infty}\Big)^q \int_{s=-\infty}^{s=\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}} \frac{1}{(2\pi)^{q/2}|\boldsymbol{D}|^{1/2}\sqrt{2\pi}} \exp\Big\{-\frac{1}{2}\Big(\frac{s}{L^{-1/2}}\Big)^2\Big\} \\
&\quad \times \exp\Big\{-\frac{1}{2}(\boldsymbol{\xi}_i - \boldsymbol{k})'\boldsymbol{M}(\boldsymbol{\xi}_i - \boldsymbol{k})\Big\}(\boldsymbol{x}'_{1ij}\boldsymbol{\beta} + \boldsymbol{z}'_{1ij}\boldsymbol{\xi}_i) d\boldsymbol{\xi}_i ds,
\end{aligned}
\tag{G.1}
$$

where

$$
\begin{aligned}
t &= s - \boldsymbol{z}_{2ik}'\boldsymbol{\xi}_i \\
\boldsymbol{\xi}_i'\boldsymbol{D}^{-1}\boldsymbol{\xi}_i + t^2 &= (\boldsymbol{\xi}_i - \boldsymbol{k})'[\boldsymbol{M}^{-1}]^{-1}(\boldsymbol{\xi}_i - \boldsymbol{k}) + \left(\frac{s}{L^{-1/2}}\right)^2 \\
k &= \boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}'s \\
\boldsymbol{M} &= \boldsymbol{D}^{-1} + \boldsymbol{z}_{2ik}\boldsymbol{z}_{2ik}' \\
L &= 1 - \boldsymbol{z}_{2ik}'\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}
\end{aligned}
$$

Next, we integrate over the random effects, which results in

$$
\begin{aligned}
&E[Y_{1ij}Y_{2ik} \leqslant c] \\
&= \int_{s=-\infty}^{s=\boldsymbol{\gamma}_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}} \frac{1}{\sqrt{2\pi|\boldsymbol{D}||\boldsymbol{M}|}} \exp\left\{-\frac{1}{2}\left(\frac{s}{L^{-1/2}}\right)^2\right\}\left(\boldsymbol{x}_{1ij}'\boldsymbol{\beta} + \boldsymbol{z}_{1ij}'\boldsymbol{k}\right)ds \\
&= \int_{s=-\infty}^{s=\boldsymbol{\gamma}_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}} \frac{1}{\sqrt{2\pi|\boldsymbol{D}||\boldsymbol{M}|}} \exp\left\{-\frac{1}{2}\left(\frac{s}{L^{-1/2}}\right)^2\right\}\left(\boldsymbol{x}_{1ij}'\boldsymbol{\beta} + \boldsymbol{z}_{1ij}'\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}s\right)ds \\
&= \int_{u=-\infty}^{u=\frac{\boldsymbol{\gamma}_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}}{L^{-1/2}}} \frac{1}{\sqrt{2\pi|\boldsymbol{D}||\boldsymbol{M}|L}} \exp\left\{-\frac{1}{2}u^2\right\}\left(\boldsymbol{x}_{1ij}'\boldsymbol{\beta} + \boldsymbol{z}_{1ij}'\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\frac{u}{L^{1/2}}\right)du,
\end{aligned}
$$

where $u = \frac{s}{L^{-1/2}}$. Integration over u results in the following equation

$$
\begin{aligned}
E[Y_{1ij}Y_{2ik} \leqslant c] = {} & \frac{1}{|\boldsymbol{D}|^{1/2}}\frac{1}{|\boldsymbol{M}|^{1/2}}\frac{1}{L^{1/2}}\boldsymbol{x}_{1ij}'\boldsymbol{\beta}\Phi(\boldsymbol{\gamma}_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}; L^{-1}) - \\
& \frac{1}{|\boldsymbol{D}|^{1/2}}\frac{1}{|\boldsymbol{M}|^{1/2}}\frac{1}{L^{3/2}}\boldsymbol{z}_{1ij}'\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\phi(\boldsymbol{\gamma}_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}; L^{-1}).
\end{aligned}
$$

Further, consider

$$|\boldsymbol{M}| \times |\boldsymbol{D}| \times L = 1$$

$$|\boldsymbol{M}| \times |\boldsymbol{D}| = L^{-1}$$

$$|\boldsymbol{M} \times \boldsymbol{D}| = (1 - \boldsymbol{z}'_{2ik}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik})^{-1}$$

$$|(\boldsymbol{D}^{-1} + \boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik}) \times \boldsymbol{D}| = 1 - \boldsymbol{z}'_{2ik}(-\boldsymbol{M} + \boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik})^{-1}\boldsymbol{z}_{2ik}$$

$$|I + \boldsymbol{D}\boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik}| = 1 - \boldsymbol{z}'_{2ik}(-\boldsymbol{M} + \boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik})^{-1}\boldsymbol{z}_{2ik}$$

$$|I + \boldsymbol{D}\boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik}| = 1 - \boldsymbol{z}'_{2ik}(-\boldsymbol{D}^{-1} - \boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik} + \boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik})^{-1}\boldsymbol{z}_{2ik}$$

$$|I + \boldsymbol{D}\boldsymbol{z}_{2ik}\boldsymbol{z}'_{2ik}| = 1 + \boldsymbol{z}'_{2ik}\boldsymbol{D}\boldsymbol{z}_{2ik}$$

The last equation uses the general property that $\text{Det}(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}') = 1 + \boldsymbol{u}'\boldsymbol{v}$(Petersen and Pedersen, 2008). As a result,

$$E[Y_{1ij}Y_{2ik} \leqslant c] = \boldsymbol{x}'_{1ij}\boldsymbol{\beta}\Phi(\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L^{-1}) - \frac{1}{L}\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\phi(\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L^{-1}). \tag{G.2}$$

Hence, the covariance equals

$$\text{Cov}[Y_{1ij}Y_{2ik} \leqslant c] = -\frac{1}{L}\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\phi(\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L^{-1}). \tag{G.3}$$

As a result, the correlation is equal to

$$\rho_{Y_{1ij},Y_{2ik}\leqslant c} = \frac{-\frac{1}{L}\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\phi(\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L^{-1})}{\left(\left(\boldsymbol{z}'_{1ij}\boldsymbol{D}\boldsymbol{z}_{1ij} + \Sigma_{1ij}\right)\Phi(\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})(1 - \Phi(\boldsymbol{\gamma}_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}))\right)^{1/2}}$$

G.1 *Standard errors*

The standard errors of the correlation can be derived by the means of the delta method. First, the correlation is Fisher z transformed. Then, the standard error of the Fisher Z transformed correlation z equals

$$SE(z) = \sqrt{\frac{\partial z}{\partial \boldsymbol{\theta}'}\text{Var}(\hat{\boldsymbol{\theta}})\frac{\partial z}{\partial \boldsymbol{\theta}}}, \tag{G.4}$$

where $\boldsymbol{\theta}$ signals the parameter vector. Note that $\boldsymbol{\theta}$ does not contain the coefficients of the continuous response(s).

Next, the gradient for a coefficient of an arbitrary predictor for the ordinal response $\boldsymbol{X}_{22}$ equals

$$\frac{\partial z}{\partial \boldsymbol{\beta}_{22}} = \frac{-1}{\rho^2-1}\frac{1}{\nu^2}\Bigg\{\nu\chi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta})\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik} + \frac{\frac{1}{L_i}\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})}{2\nu}\big(\boldsymbol{z}'_{1ij}\boldsymbol{D}\boldsymbol{z}_{1ij} + \Sigma_{1ij}\big) \Big(- \chi\Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}) + \chi(1 - \Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}))\Big)\Bigg\},$$

where ρ equals the non-transformed correlation between Y_{1ij} and $Y_{2ik} \leqslant c$ and

$$\chi = -X_{2ik}\phi(\gamma_c - \boldsymbol{x}'_{22ik}\boldsymbol{\beta}; \frac{1}{L_i}),$$
$$\nu = \Bigg(\big(\boldsymbol{z}'_{1ij}\boldsymbol{D}\boldsymbol{z}_{1ij} + \Sigma_{1ij}\big)\Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})(1 - \Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}))\Bigg)^{1/2}.$$

Further, the gradient for the threshold value γ_c equals

$$\frac{\partial z}{\partial \gamma_c} = \frac{-1}{\rho^2-1}\frac{1}{\nu^2}\Bigg\{\nu(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta})\phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; \frac{1}{L_i})\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik} + \frac{\frac{1}{L_i}\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})}{2\nu}\big(\boldsymbol{z}'_{1ij}\boldsymbol{D}\boldsymbol{z}_{1ij} + \Sigma_{1ij}\big) \Big(- \phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})\Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}) + \phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})(1 - \Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}))\Big)\Bigg\}.$$

In addition, the gradient for τ, an arbitrary component of the random effects variance-covariance matrix equals

$$\frac{\partial z}{\partial \tau} = \frac{-1}{\rho^2-1}\frac{1}{\nu^2}\Bigg[\frac{\nu\phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; \frac{1}{L_i})}{L_i}\Bigg(\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\Bigg\{\frac{L_i^*}{L_i} + \frac{L_i^*(L_i(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta})^2 - 1)}{2L_i}\Bigg\} - \boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{D}^{-1}\boldsymbol{D}^*\boldsymbol{D}^{-1}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\Bigg) + \frac{1}{L_i}\boldsymbol{z}'_{1ij}\boldsymbol{M}^{-1}\boldsymbol{z}_{2ik}\phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})\frac{1}{2\nu}\Bigg(\boldsymbol{z}'_{1ij}\boldsymbol{D}^*\boldsymbol{z}_{1ij}\Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})(1 - \Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1})) + \big(\boldsymbol{z}'_{1ij}\boldsymbol{D}\boldsymbol{z}_{1ij} + \Sigma_{1ij}\big)\Big(- G\Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}) + G(1 - \Phi(\gamma_c - \boldsymbol{x}'_{2ik}\boldsymbol{\beta}; L_i^{-1}))\Big)\Bigg)\Bigg],$$

where

$$\boldsymbol{D}^* = \frac{\partial \boldsymbol{D}}{\partial \tau},$$

$$L_i^* = -\boldsymbol{z}_{2ik}' \boldsymbol{M}^{-1} \boldsymbol{D}_{lm}^{-1} \boldsymbol{D}^* \boldsymbol{D}_{lm}^{-1} \boldsymbol{M}^{-1} \boldsymbol{z}_{2ik},$$

$$G = \frac{L_i^*(\gamma_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta})}{2\sqrt{L_i}} \phi\left(\sqrt{L_i}(\gamma_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta})\right).$$

Lastly, the gradient with respect to σ_{1ij}^2 equals

$$\frac{\partial z}{\partial \sigma_{1ij}^2} = \frac{1}{\rho^2 - 1} \frac{\frac{-1}{L} \boldsymbol{z}_{1ij}' \boldsymbol{M}^{-1} \boldsymbol{z}_{2ik} \phi(\gamma_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}; L_i^{-1}) \Phi(\gamma_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}; L_i^{-1})(1 - \Phi(\gamma_c - \boldsymbol{x}_{2ik}'\boldsymbol{\beta}; L_i^{-1}))}{2\nu^3},$$

H. Case study: Post-operative functioning

H.1 *Point-Biserial Correlations*

Table 6. *Point-Biserial Correlations between ADL (higher: lower functioning) and MMSE (cognitive impairment)*

Panel A: Correlations between ADL and the event of having severe impairment.					
	Time(Impairment)				
Time (ADL)	1	3	5	8	12
1	0.57 [0.37;0.72]	0.50 [0.27;0.67]	0.62 [0.43;0.75]	0.70 [0.53;0.82]	0.54 [0.27;0.73]
5	0.62 [0.43;0.76]	0.53 [0.32;0.70]	0.62 [0.43;0.75]	0.76 [0.62;0.86]	0.61 [0.36;0.78]
12	0.74 [0.55;0.86]	0.58 [0.31;0.76]	0.65 [0.42;0.80]	0.74 [0.55;0.86]	0.64 [0.40;0.80]
Panel B: Correlations between ADL and the event of having impairment.					
	Time(Impairment)				
Time (ADL)	1	3	5	8	12
1	0.65 [0.48;0.78]	0.66 [0.49;0.79]	0.58 [0.38;0.73]	0.64 [0.45;0.78]	0.64 [0.40;0.80]
5	0.64 [0.46;0.77]	0.68 [0.51;0.80]	0.63 [0.44;0.76]	0.74 [0.59;0.85]	0.78 [0.61;0.88]
12	0.72 [0.53;0.85]	0.71 [0.50;0.84]	0.76 [0.58;0.86]	0.78 [0.62;0.88]	0.80 [0.64;0.89]

H.2 *SAS code for the joint model*

Obs	ID	SEX	AGE	ADLTOT1	ADLTOT5	ADLTOT12	MMSE1	MMSE5	MMSE8	MMSE12
1	1	1	74	20	9	7	28	28	26	25
2	2	2	67	16	11	.	25	23	27	.
3	3	1	67	13	9	.	26	29	27	.
4	4	2	88	17	14	15	25	27	27	25
5	5	2	87	17	17	.	16	19	17	.

Fig. 2. First five observations in the dataset.

The first five entries from the dataset are illustrated in Figure 2. Each subject occupies a single row within the dataset. To facilitate data analysis, we intend to restructure the data, converting it into a ‘long’ format where each observation corresponds to a separate row. Achieving this transformation can be accomplished using the subsequent SAS code snippet:

```
DATA g.long_s;
  SET g.wide_s;
  time = 1 ;
  ADLTOT=ADLTOT1;
  MMSE=MMSE1;
  OUTPUT ;
  time = 3;
  ADLTOT=.;
  MMSE=MMSE3;
  OUTPUT ;
  time = 5 ;
    ADLTOT=ADLTOT5;
   MMSE=MMSE5;
  OUTPUT ;
   time = 8 ;
     ADLTOT=.;
   MMSE=MMSE8;
  OUTPUT ;
   time = 12 ;
    ADLTOT=ADLTOT12;
   MMSE=MMSE12;
  OUTPUT ;
  keep ID SEX AGE time ADLTOT MMSE;
RUN;
```

Obs	ID	SEX	AGE	time	ADLTOT	MMSE
1	1	1	74	1	20	28
2	1	1	74	3	.	28
3	1	1	74	5	9	28
4	1	1	74	8	.	26
5	1	1	74	12	7	25
6	2	2	67	1	16	25
7	2	2	67	3	.	25
8	2	2	67	5	11	23
9	2	2	67	8	.	27
10	2	2	67	12	.	.

Fig. 3. Sample of the dataset in a ‘long’ format.

The result is presented in Figure 3. We will now transform the MMSE variable in a clinically relevant ordinal variable.

```
data g.long_s;
set g.long_s;
length impairment $6;
if mmse>23 then impairment='2';
else if mmse>17 then impairment='1';
else if mmse>0 then impairment='0';
run;
```

Subsequently, another round of data transformation is conducted. In order to fit a joint model, the data needs to have a single line for each measurement for each response. Furthermore, the identification of the appropriate link function and distribution for each observation is also required. Next, dummy variables were created for both gender and time. Additionally, the time variable was divided by 100 with the intention of increasing the variance of the random effects. This facilitates achieving convergence in the modeling process.

```
data g.analysis_s;
set g.long_s;
length distvar $11;
length response 8;
length linkvar $11;
length var $20;
response = ADLTOT;
var='ADLTOT';
distvar    = "Normal";
linkvar = "IDEN";
output;
response = impairment;
var='impairment';
distvar    = "multinomial";
linkvar =  "CPROBIT";
output;
keep ID SEX AGE TIME distvar response var linkvar;
run;

data g.analysis_s;
set g.analysis_s;
SEX=SEX-1;
if time=5 then time_5=1; else time_5=0;
if time=12 then time_12=1; else time_12=0;
time_d100=time/100;
run;
```

Obs	ID	SEX	AGE	time	distvar	response	linkvar	var	time_5	time_12	time_d100
1	1	0	74	1	Normal	20	IDEN	ADLTOT	0	0	0.01
2	1	0	74	1	multinomial	2	CPROBIT	impairment	0	0	0.01
3	1	0	74	3	Normal	.	IDEN	ADLTOT	0	0	0.03
4	1	0	74	3	multinomial	2	CPROBIT	impairment	0	0	0.03
5	1	0	74	5	Normal	9	IDEN	ADLTOT	1	0	0.05
6	1	0	74	5	multinomial	2	CPROBIT	impairment	1	0	0.05
7	1	0	74	8	Normal	.	IDEN	ADLTOT	0	0	0.08
8	1	0	74	8	multinomial	2	CPROBIT	impairment	0	0	0.08
9	1	0	74	12	Normal	7	IDEN	ADLTOT	0	1	0.12
10	1	0	74	12	multinomial	2	CPROBIT	impairment	0	1	0.12
11	2	1	67	1	Normal	16	IDEN	ADLTOT	0	0	0.01
12	2	1	67	1	multinomial	2	CPROBIT	impairment	0	0	0.01
13	2	1	67	3	Normal	.	IDEN	ADLTOT	0	0	0.03
14	2	1	67	3	multinomial	2	CPROBIT	impairment	0	0	0.03
15	2	1	67	5	Normal	11	IDEN	ADLTOT	1	0	0.05

Fig. 4. Sample of the dataset after data manipulation.

The result is shown in Figure 4. Finally, we fit the joint model with `PROC NLMIXED`. Starting values are obtained from the univariate models and the joint model with a looser convergence criterion.

```
proc nlmixed data=g.analysis_s qpoints=20 maxiter=1000
maxfunc=10000 technique=quanew cov;
parms
gamma1 = -20.2253
gamma2  =       -17.716
beta1_time=     0.04737
beta1_fem  =    -0.4749
beta1_age  =    -0.2298
sigma2     =    3.038746
beta2_1    =    3.2973
beta2_time5 =   -2.6919
beta2_time12=   -3.6249
beta2_fem  =    -1.6175
beta2_age  =    0.2048
tau1     =     9.5815
tau12    =     2.9075
tau2     =     62.9913
tau3     =     7.2346
tau34    =     10.4628
tau4     =     695
tau31    =     -5.7428
tau32    =     2.353
```

```
tau41   =    -7.11
tau42   =    -1.21891
 ;
eta = beta1_time*time+beta1_fem*SEX+beta1_age*AGE+
a+b*time_d100;
if var='impairment' then do;
if response =0 then do;
lik = cdf('NORMAL',(gamma1-eta));
end;
if response =1 then do;
lik = cdf('NORMAL',(gamma2-eta)) -
cdf('NORMAL',(gamma1-eta));
end;
if response =2 then do;
lik = 1 -cdf('NORMAL',(gamma2-eta));
end;
ll = log(lik);
end;
if var='ADLTOT' then do;
mean = c+d*time_d100+
beta2_1+beta2_time5*time_5+beta2_time12*time_12+beta2_fem*SEX+beta2_age*AGE;
dens = -0.5*log(3.14) - log(sqrt(sigma2)) -
0.5*(response-mean)**2/(sigma2);
ll = dens;
end;
model response ~ general(ll);
random a b c d~
    normal([0,0,0,0],[tau1,tau12,tau2,tau31,tau32,tau3,tau41,tau42,tau34,tau4])
subject = ID;
where var='ADLTOT' or var='impairment';
ods output parameterestimates=g.parms_joint CovMatParmEst=g.covb;
run;
```